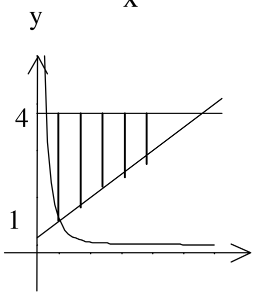
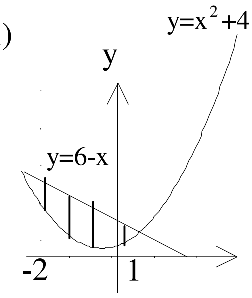
<u>ΛΥΣΕΙΣ ΑΣΚΗΣΕΩΝ</u>
<u>ΚΕΦΑΛΑΙΟ 9^{o}</u>

9.1 (i) $y=\frac{1}{x^2}$, y=x, y=4



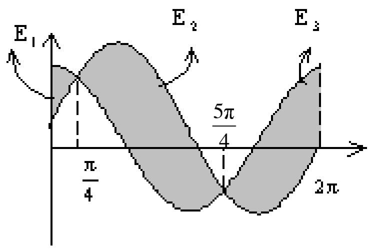
$$E=\int_1^4\left(y-\frac{1}{\sqrt{y}}\right)dy=\left.\frac{y^2}{2}-2\sqrt{y}\right]_1^4=\frac{11}{2}$$

(ii)



$$E=\int_{-2}^1\left[(6-x)-\left(x^2+4\right)\right]dx=\left[2x-\frac{1}{x^2}-\frac{x^3}{3}\right]_{-2}^1=\frac{9}{2}$$

(iii)



$$E=E_1+E_2+E_3$$

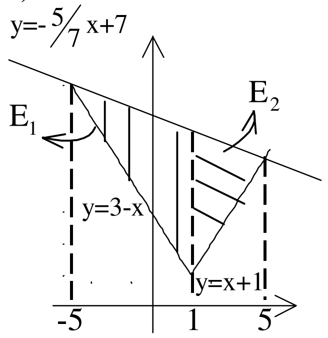
$$E_1=\int_0^{\pi/4}(cosx-sinx)dx=\left.sinx+cosx\right]_0^{\pi/4}=\sqrt{2}-1$$

$$E_2=\int_{\pi/4}^{5\pi/4}(sinx-cosx)dx=\left.-cosx-sinx\right]_{\pi/4}^{5\pi/4}=2\sqrt{2}$$

$$E_3=\int_{5\pi/4}^{2\pi}(cosx-sinx)dx=\left.sinx+cosx\right]_{5\pi/4}^{2\pi}=1+\sqrt{2}$$

$$E=4\sqrt{2}$$

(iv)



$$E=E_1+E_2$$

$$E_1=\int_{-5}^1\left[\left(-\frac{x}{5}+7\right)-(3-x)\right]dx=\int_{-5}^1\left(\frac{4x}{5}+4\right)dx$$

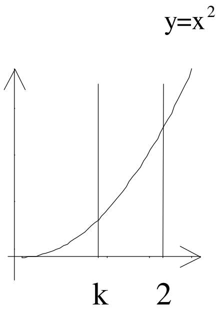
$$=\left.\frac{2x^2}{5}+4x\right]_{-5}^1=14\frac{2}{5}$$

$$E_2=\int_1^5\left[\left(-\frac{x}{5}+7\right)-(x+1)\right]dx=\int_1^5\left(-\frac{6x}{5}+6\right)dx$$

$$=\left.-\frac{3x^2}{5}+6x\right]_1^5=9\frac{3}{5}$$

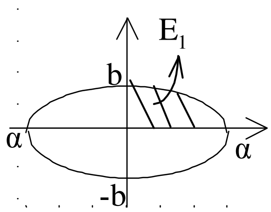
$$E=14\frac{2}{5}+9\frac{3}{5}=24$$

9.2 Πρέπει: $\int_0^k x^2dx = \int_k^2 x^2dx$



$$\Leftrightarrow \left.\frac{x^3}{3}\right|_0^k = \left.\frac{x^3}{3}\right|_k^2 \Leftrightarrow \frac{k^3}{3}=\frac{8-k^3}{3} \Rightarrow k^3=4 \Rightarrow k=\sqrt[3]{4}$$

9.3 Λόγω συμμετρίας: $E=4E_1$



$$E=4\int_0^{α} \frac{b}{α}\sqrt{α^2-x^2}\,dx=\frac{4b}{α}\int_0^{α}\sqrt{α^2-x^2}\,dx=\frac{4b}{α}I$$

Για τον υπολογισμό του I έχουμε:

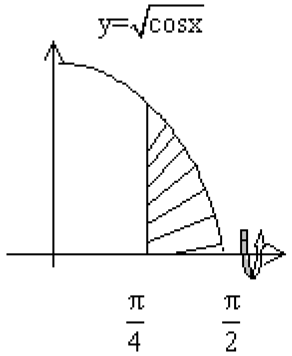
$x=α\sin θ \Rightarrow dx=α\cos θ$

$x=0 \to θ=0,\ x=α \to θ=\frac{\pi}{2}$

$$I=\int_0^{\pi/2} α\cos θ\, α\sin θ\, dθ= α^2\int_0^{\pi/2}\cos^2θ\, dθ α^2\int_0^{\pi/2}\frac{1+\cos 2θ}{2}\, dθ=α^2\left[\frac{θ}{2}+\frac{\sin 2θ}{4}\right]_0^{\pi/2}=\frac{α^2\pi}{4}$$

Άρα: $E=\frac{4b}{α}I=\frac{4b}{α}\frac{\pi α^2}{4}=\pi α b$

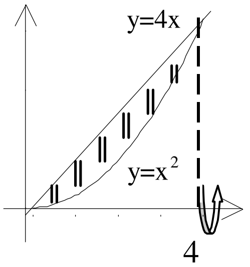
9.4 (i) $y=\cos x,\ x=\frac{\pi}{4},\ x=\frac{\pi}{2},\ y=0$



$$V=\pi\int_{\pi/4}^{\pi/2}\cos x\,dx= \pi\left[\sin x\right]_{\pi/4}^{\pi/2}=\frac{\pi\left(2-\sqrt{2}\right)}{2}$$

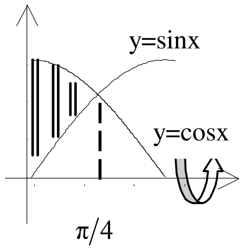
(ii) $y=x^2, y=4x$

$$V=\pi\int_{\pi/4}^{\pi/2}\left[\left(4x\right)^2-\left(x^2\right)^2\right]dx=\pi\int_{\pi/4}^{\pi/2}\left[16x^2-x^4\right]dx$$



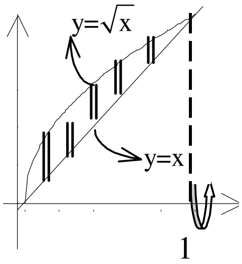
$$=\pi\left[\frac{16}{3}x^3-\frac{1}{5}x^5\right]_{\pi/4}^{\pi/2}=\frac{2048}{15}\pi$$

9.4 (iii) $y=\sin x,\ y=\cos x,\ x=0,\ x=\dfrac{\pi}{4}$



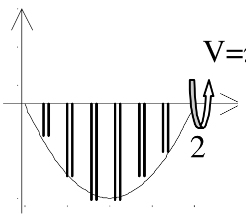
$$V=\pi\int_0^{\pi/4}\left(\cos^2x-\sin^2x\right)dx=\pi\int_0^{\pi/4}\cos 2x\,dx=\pi\left[\frac{\sin 2x}{2}\right]_0^{\pi/4}=\frac{\pi}{2}$$

(iv) $y=\sqrt{x},\ y=x$



$$V=\pi\int_0^1\left[\left(\sqrt{x}\right)^2-x^2\right]dx=\pi\int_0^1\left(x-x^2\right)dx=\pi\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=\frac{\pi}{6}$$

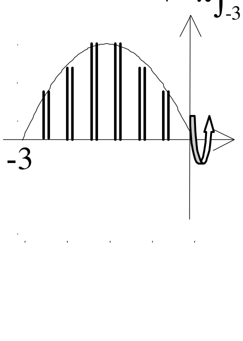
(v) $y=x^2-2x,\ y=0$



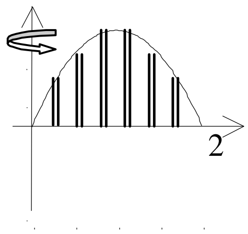
$$V=\pi\int_0^2\left(x^2-2x\right)^2dx=\pi\int_0^2\left(x^4-4x^3+4x^2\right)dx=\pi\left[\frac{x^5}{5}-x^4+\frac{4x^3}{3}\right]_0^2=\frac{16\pi}{15}$$

(vi) $y=-3x-x^2,\ y=0$

$$V=\pi\int_{-3}^0\left(-3x-x^2\right)^2dx=\pi\int_{-3}^0\left(9x^2+6x^3+x^4\right)dx=\pi\left[3x^3+\frac{3}{2}x^4+\frac{x^5}{5}\right]_{-3}^0=\frac{81\pi}{10}$$



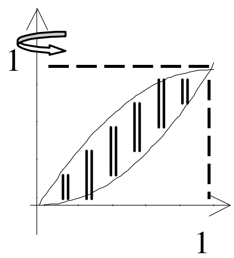
9.5 (i) $y=2x-x^2,\ y=0$



$$V=2\pi\int_0^2 xf(x)dx=2\pi\int_0^2 x\left(2x-x^2\right)dx=2\pi\left[\frac{2}{3}x^3-\frac{x^4}{4}\right]_0^2=\frac{8}{3}\pi$$

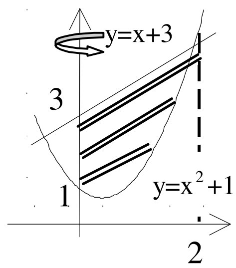
(ii) $y=x^2,\ x=y^2$

$$V=2\pi\int_0^1 x\left[\sqrt{x}-x^2\right]dx=2\pi\int_0^1\left(x^{3/2}-x^3\right)dx=2\pi\left[\frac{2}{5}x^{5/2}-\frac{x^4}{4}\right]_0^1=\frac{3}{10}\pi$$



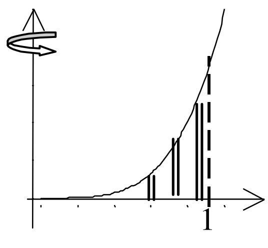
(iii) $y=x^2+1,\ y=x+3,\ x=0$

$$V=2\pi\int_\alpha^\beta x\left(y_1-y_2\right)dx$$



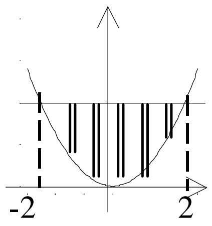
$$V=2\pi\int_0^2 x\left[\left(x+3\right)-\left(x^2+1\right)\right]dx=2\pi\int_0^2\left(x^2+2x-x^3\right)dx=2\pi\left[\frac{x^3}{3}+x^2-\frac{x^4}{4}\right]_0^2=\frac{16}{3}\pi$$

(iv) $y=x^4,\ x=1,\ y=0$



$$V=2\pi\int_0^1 xydx=2\pi\int_0^1 x\left(x^4\right)dx=2\pi\left[\frac{x^6}{6}\right]_0^1=\frac{\pi}{3}$$

9.6



(i)Περιστροφή γύρω από τον άξονα των x :

$$V=\pi\int_{-2}^{2}\left[4^2-\left(x^2\right)^2\right]dx=\pi\int_{-2}^{2}\left(16-x^4\right)dx=\pi\left[16x-\frac{x^5}{5}\right]_{-2}^{2}=\frac{256}{5}\pi$$

(ii) Περιστροφή γύρω από τη ευθεία y=4:

$$V=\pi\int_{-2}^{2}\left(x^2-4\right)^2dx=\pi\int_{-2}^{2}\left(16-8x^2+x^4\right)dx=\pi\left[16x-\frac{8}{3}x^3+\frac{x^5}{5}\right]_{-2}^{2}=\frac{512}{15}\pi$$

(iii) Περιστροφή γύρω από τον άξονα των y :

$$V=\pi\int_{0}^{4}\left(\sqrt{y}\right)^2dy=\pi\int_{0}^{4}ydx=\pi\left[\frac{y^2}{2}\right]_{0}^{4}=8\pi$$

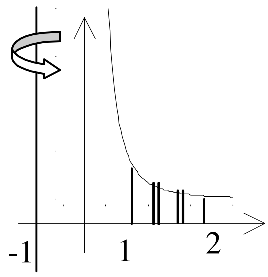
(iv) Περιστροφή γύρω από την ευθεία y= -1:

$$V=\pi\int_{-2}^{2}\left[(4+1)^2-\left(x^2+1\right)^2\right]dx=\pi\int_{-2}^{2}\left(24-x^4-2x^2\right)dx=\pi\left[24x-\frac{x^5}{5}-\frac{2}{3}x^3\right]_{-2}^{2}=\frac{1088}{15}\pi$$

(v) Περιστροφή γύρω από τη ευθεία x=2

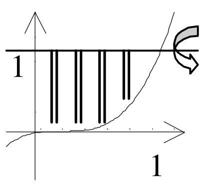
$$V=2\pi\int_{-2}^{2}(2-x)\left(4-x^2\right)dx=2\pi\int_{-2}^{2}\left(8-2x^2-4x+x^3\right)dx=2\pi\left[8x-\frac{2x^3}{3}-2x^2+\frac{x^4}{4}\right]_{-2}^{2}=\frac{128}{3}\pi$$

9.7 $y=\frac{1}{x^3}$, x=1, x=0. Γύρω από την ευθεία x=–1



$$V=2\pi\int_{1}^{2}(x+1)ydx=2\pi\int_{1}^{2}(x+1)\frac{1}{x^3}dx=2\pi\int_{1}^{2}\left(\frac{1}{x^2}+\frac{1}{x^3}\right)dx=2\pi\left[-\frac{1}{x}-\frac{1}{2x^2}\right]_{1}^{2}=\frac{7}{4}\pi$$

9.8 $V=\pi\int_{0}^{1}\left(1-y^2\right)dx=\pi\int_{0}^{1}\left(1-x^3\right)^2dx=\pi\int_{0}^{1}\left(1-2x^3+x^6\right)dx=\pi\left[x-\frac{x^4}{2}+\frac{x^7}{7}\right]_{0}^{1}=\frac{9}{14}\pi$



9.9 (i) $y=\frac{1}{3}\left(x^2+2\right)^{3/2} \Rightarrow y'=x\sqrt{x^2+2}$

$$L=\int_0^3\sqrt{1+(y')^2}dx=\int_0^3\sqrt{1+x^2\left(x^2+2\right)}dx=\int_0^3\sqrt{\left(1+x^2\right)^2}dx=\int_0^3\left(1+x^2\right)dx=\left[x+\frac{x^3}{3}\right]_0^3=12$$

(ii) $9x^2=4y^3 \Rightarrow 18xx'=12y^2 \Rightarrow x'=\frac{2y^2}{3x} \Rightarrow (x')^2=\frac{4y^4}{9x^2}=\frac{4y^4}{4y^3}=y$

$$L=\int_0^3\sqrt{1+(x')^2}dy=\int_0^3\sqrt{1+y}dy=\left[\frac{2}{3}(1+y)^{3/2}\right]_0^3=\frac{14}{3}$$

(iii) $y=\frac{x^3}{3}+\frac{1}{4x} \Rightarrow y'=x^2-\frac{1}{4x^2}$

$$L=\int_1^3\sqrt{1+(y')^2}dx=\int_1^3\sqrt{1+\left(x^2-\frac{1}{4x^2}\right)^2}dx=\int_1^3\sqrt{1+x^4-\frac{1}{2}+\frac{1}{16x^4}}dx=\int_1^3\sqrt{x^4+\frac{1}{2}+\frac{1}{16x^4}}dx$$

$$=\int_1^3\sqrt{\left(x^2+\frac{1}{4x^2}\right)^2}dx=\int_1^3\left(x^2+\frac{1}{4x^2}\right)dx=\left[\frac{x^3}{3}-\frac{1}{4x}\right]_1^3=\frac{53}{6}$$

(iv) $(y+1)^2=4x^3 \Rightarrow 2(y+1)y'=12x^2 \Rightarrow y'=\frac{6x^2}{y+1} \Rightarrow (y')^2=\frac{36x^4}{(y+1)^2}=\frac{36x^2}{4x^3}=9x$

$$\Rightarrow L=\int_0^1\sqrt{1+9x}dx=\left[\frac{2}{3}\frac{(1+9x)^{3/2}}{9}\right]_0^1=\frac{2\left(10\sqrt{10}-1\right)}{27}$$

9.10 (i) $y=\sqrt{x} \Rightarrow y'=\frac{1}{2\sqrt{x}}$

$$S=2\pi\int_1^4 y\sqrt{1+(y')^2}dx=2\pi\int_1^4\sqrt{x}\sqrt{1+\frac{1}{4x}}dx=\pi\int_1^4(1+4x)^{1/2}\,dx$$

$$=\pi\left[\frac{2}{3}\frac{(1+4x)^{3/2}}{4}\right]_1^4=\pi\frac{17\sqrt{17}-5\sqrt{5}}{6}$$

(ii) $y=\frac{x^3}{3}+\frac{1}{4x} \Rightarrow y'=x^2-\frac{1}{4x^2}$

$$S=\int_1^2\left(\frac{x^3}{3}+\frac{1}{4x}\right)\sqrt{1+\left(x^2-\frac{1}{4x^2}\right)^2}dx$$

$$=2\pi\int_1^2\left(\frac{x^3}{3}+\frac{1}{4x}\right)\sqrt{1+x^4-\frac{1}{2}+\frac{1}{16x^4}}dx=2\pi\int_1^2\left(\frac{x^3}{3}+\frac{1}{4x}\right)\sqrt{x^4+\frac{1}{2}+\frac{1}{16x^4}}dx$$

$$=2\pi\int_1^2\left(\frac{x^3}{3}+\frac{1}{4x}\right)\sqrt{\left(x^2+\frac{1}{4x^2}\right)^2}dx=2\pi\int_1^2\left(\frac{x^3}{3}+\frac{1}{4x}\right)\left(x^2+\frac{1}{4x^2}\right)dx$$

$$=2\pi\int_1^2\left(\frac{x^5}{3}+\frac{1}{3}x+\frac{1}{16x^3}\right)dx=\ldots=\frac{515}{64}\pi$$

9.10 (iii) $y=x^3 \Rightarrow y'=3x^2$

$$S=2\pi\int_0^1 x^3\sqrt{1+9x^4}dx=\frac{1}{18}\pi\int_0^1 36x^3\sqrt{1+9x^4}dx=\frac{1}{18}\pi\left[\frac{2\left(1+9x^4\right)^{3/2}}{3}\right]_1^2$$

$$=\frac{\pi}{27}\left(145\sqrt{145}-10\sqrt{10}\right)$$

(iv) $y=\sqrt{2x-x^2} \Rightarrow y'=\frac{2-2x}{\sqrt{2x-x^2}}=\frac{1-x}{\sqrt{2x-x^2}}$

$$S=2\pi\int_{1/2}^1 \sqrt{2x-x^2}\sqrt{1+\left(\frac{1-x}{\sqrt{2x-x^2}}\right)^2}dx=2\pi\int_{1/2}^1\sqrt{2x-x^2}\sqrt{1+\frac{1+x^2-2x}{2x-x^2}}dx$$

$$=2\pi\int_{1/2}^1 \cancel{\sqrt{2x-x^2}}\,\frac{\sqrt{2x-x^2+1+x^2-2x}}{\cancel{\sqrt{2x-x^2}}}dx=2\pi\int_{1/2}^1 1dx=2\pi\left[x\right]_{1/2}^1=\pi$$

9.11 (i) $y=\frac{2}{3}x^{3/2}-\frac{1}{2}x^{1/2} \Rightarrow y'=x^{1/2}-\frac{1}{4x^{1/2}}$

$$S=2\pi\int_0^9(9-x)\sqrt{1+\left(x^{1/2}-\frac{1}{4x^{1/2}}\right)^2}dx=2\pi\int_0^9(9-x)\sqrt{1+x-\frac{1}{2}+\frac{1}{16x}}dx$$

$$=2\pi\int_0^9(9-x)\sqrt{\frac{1}{2}+x+\frac{1}{16x}}dx=2\pi\int_0^9(9-x)\sqrt{\left(x^{1/2}+\frac{1}{4x^{1/2}}\right)^2}dx=2\pi\int_0^9(9-x)\left(x^{1/2}+\frac{1}{4x^{1/2}}\right)dx$$

$$=2\pi\int_0^9\left(\frac{35}{4}x^{1/2}+\frac{9}{4x^{1/2}}-x^{3/2}\right)dx=\frac{738}{5}\pi$$

9.12 $y^2=4x \Rightarrow y=2\sqrt{x} \Rightarrow y'=\frac{1}{\sqrt{x}}$

$$\left|\overline{OA}\right|=\sqrt{1^2+2^2}=\sqrt{5}$$

$$\overset{\frown}{OA}=\int_0^1\sqrt{1+\frac{1}{x}}dx=...=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$$

Άρα μήκος τόξου μεικτόγραμμου : $\sqrt{5}+\sqrt{2}+\ln\left(1+\sqrt{2}\right)$

9.13 (i) $y=\cosh 2x$, $y=\sinh 2x$, $x=0$, $x=5$

$$V=\pi\int_0^5\left(\cosh^2 2x-\sinh^2 2x\right)dx=\pi\int_0^5 dx=5\pi$$

(ii) $y=\operatorname{sech}x$, $y=0$, $x=0$, $x=\ln 2$

$$V=\pi\int_0^{\ln 2}\operatorname{sech}^2 x dx=\pi\left[\tanh x\right]_0^{\ln 2}=\pi\tanh\left(\ln 2\right)=\pi\frac{e^{\ln 2}-e^{-\ln 2}}{e^{\ln 2}+e^{-\ln 2}}=\frac{3\pi}{5}$$

(iii) $y=e^x$. $y=0$, $x=0$, $x=2$

$$V=\pi\int_0^2 e^{2x}dx=\pi\left[\frac{1}{2}e^{2x}\right]_0^2=\frac{\pi}{2}\left(e^4-1\right)$$

9.14 $y=\dfrac{1}{1+x^4}$, y=0, x=1, x=b

$$V=2\pi\int_1^b xydx=2\pi\int_1^b \frac{x}{1+x^4}dx=\pi\left[\tan^{-1}x^2\right]_1^b=\pi\left[\tan^{-1}b-\frac{\pi}{4}\right]$$

$$\lim_{b\to\infty}V=\lim_{b\to\infty}\pi\left[\tan^{-1}b-\frac{\pi}{4}\right]=\pi\left[\frac{\pi}{2}-\frac{\pi}{4}\right]=\frac{\pi^2}{4}$$

9.16 $y=\dfrac{2}{\cos2x-\sin2x}=\dfrac{2\text{cosec}2x}{\cot2x-1}$

$$V=\pi\int_{\pi/4}^{5\pi/12}\left[f(x)\right]^2dx=\pi\int_{\pi/4}^{5\pi/12}\frac{4\text{cosec}^2 2x}{(\cot2x-1)^2}dx=\pi\left[\frac{2}{\cot2x-1}\right]_{\pi/4}^{5\pi/12}=\pi\left(\frac{2}{-\sqrt{3}-1}-\frac{2}{0-1}\right)$$

$$=\pi\left(\frac{2}{-\sqrt{3}-1}+2\right)$$