ΛΥΣΕΙΣ ΑΣΚΗΣΕΩΝ
ΚΕΦΑΛΑΙΟ 8^{o}

8.1 (i) $\int \frac{\ln x}{\sqrt{x}}dx=2\sqrt{x}\ln x-\int \frac{2\sqrt{x}}{x}dx=2\sqrt{x}\ln x-2\int \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\ln x-4\sqrt{x}+c$

(ii) $\int x\tan^{-1}x\,dx=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int x^2\frac{1}{1+x^2}\,dx=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\,dx$

$=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\,dx=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int \frac{\cancel{1+x^2}}{\cancel{1+x^2}}\,dx+\frac{1}{2}\int \frac{1}{1+x^2}dx$

$=\frac{1}{2}\left[x^2\tan^{-1}x-x+\tan^{-1}x\right]+c$

(iii) $I=\int e^{-3x}\sin 3x\,dx=-\frac{1}{3}e^{-3x}\sin 3x+\frac{1}{3}\int e^{-3x}3\cos 3x\,dx$

$=-\frac{1}{3}e^{-3x}\sin 3x-\frac{1}{3}e^{-3x}\cos 3x+\frac{1}{3}\int e^{-3x}(-3)\sin 3x\,dx=-\frac{1}{3}e^{-3x}\sin 3x-\frac{1}{3}e^{-3x}\cos 3x-I$

$\Rightarrow 2I=-\frac{1}{3}\left(e^{-3x}\sin 3x+e^{-3x}\cos 3x\right)\Rightarrow I=-\frac{1}{6}\left(e^{-3x}\sin 3x+e^{-3x}\cos 3x\right)+c$

(iv) $I=\int \cos(\ln x)dx$

Θέτουμε : $u=\ln x \Rightarrow \frac{du}{dx}=\frac{1}{x}\Rightarrow e^u du=dx$

$I=\int e^u\cos u\,du=e^u\cos u+\int e^u\sin u\,du=e^u\cos u+e^u\sin u-\int e^u\cos u\,du=e^u\cos u+e^u\sin u-I$

$\Rightarrow 2I=e^u\cos u+e^u\sin u\Rightarrow \int \cos(\ln x)dx=\frac{1}{2}\left[x\cos(\ln x)+x\sin(\ln x)\right]+c$

(v) $\int \sin^{-1}x\,dx=x\sin^{-1}x-\int \frac{x}{\sqrt{1-x^2}}dx=x\sin^{-1}x+\sqrt{1-x^2}+c$

(vi) $\int \frac{xe^x}{(x+1)^2}dx=\frac{e^x}{x+1}+c$

(vii) $\int \frac{\ln x}{x^2}dx=-\frac{1}{x}\ln x+\int \frac{1}{x^2}dx=-\frac{1}{x}\ln x-\frac{1}{x}+c$

(viii) $\int \frac{xdx}{x^2+6x+13}=\frac{1}{2}\int \frac{2xdx}{x^2+6x+13}=\frac{1}{2}\int \frac{2x+6-6}{x^2+6x+13}dx$

$=\frac{1}{2}\int \frac{2x+6}{x^2+6x+13}dx-\frac{1}{2}\int \frac{6}{(x+3)^2+4}dx=\frac{1}{2}\ln\left(x^2+6x+13\right)-3\int \frac{1}{4}\frac{dx}{\left(\frac{x+3}{2}\right)^2+1}$

$=\frac{1}{2}\ln\left(x^2+6x+13\right)-\frac{3}{4}2\tan^{-1}\left(\frac{x+3}{2}\right)\int \frac{1}{4}\frac{dx}{\left(\frac{x+3}{2}\right)^2+1}$

$=\frac{1}{2}\ln\left(x^2+6x+13\right)-\frac{3}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+c$

8.2 $\int_{-1}^{1} x f''(x)dx = \int_{-1}^{1} x d\left(f'(x)\right) = xf'(x)\Big|_{-1}^{1} - \int_{-1}^{1} f'(x)dx$

$= \left[f'(1)-(-1)f'(-1)\right] - f(x)\Big|_{-1}^{1} = f'(1)+f'(-1)-f(1)+f(-1)$

8.3 (i) $\int_{0}^{2\pi} \sin(mx)\cos(nx)dx = \frac{1}{2}\int_{0}^{2\pi}\left\{\sin\left[(m+n)x\right]+\sin\left[(m-n)x\right]\right\}dx$

$= \left\{-\frac{\cos\left[(m+n)x\right]}{m+n} - \frac{\cos\left[(m-n)x\right]}{m-n}\right\}\Big|_{0}^{2\pi} = -\frac{\cos\left[2(m+n)\pi\right]}{m+n} - \frac{\cos\left[2(m-n)\pi\right]}{m-n} + 1 + 1$

$= -1-1+2=0$

Επειδή : $\cos 2k\pi = \cos 0 = 1, \forall$ ακέραιο k

(ii) $\int_{0}^{2\pi} \cos(mx)\cos(nx)dx = \frac{1}{2}\int_{0}^{2\pi}\left\{\cos\left[(m+n)x\right]+\cos\left[(m-n)x\right]\right\}dx$

$= \frac{1}{2}\left\{\frac{\sin\left[(m+n)x\right]}{m+n} + \frac{\sin\left[(m-n)x\right]}{m-n}\right\}\Big|_{0}^{2\pi} = \frac{1}{2}\frac{\sin\left[2(m+n)\pi\right]}{m+n} + \frac{1}{2}\frac{\sin\left[2(m-n)\pi\right]}{m-n} - \frac{1}{2}\frac{\sin 0}{m+n} - \frac{1}{2}\frac{\sin 0}{m-n} = 0$

Επειδή : $\sin 2k\pi = \sin 0 = 0, \forall$ ακέραιο k

(iii) $\int_{0}^{2\pi} \sin(mx)\sin(nx)dx = \frac{1}{2}\int_{0}^{2\pi}\left\{\cos\left[(m-n)x\right]-\cos\left[(m+n)x\right]\right\}dx$

$= \frac{1}{2}\left\{\frac{\sin\left[(m-n)x\right]}{m-n} - \frac{\sin\left[(m+n)x\right]}{m+n}\right\}\Big|_{0}^{2\pi}$

$= \frac{1}{2}\frac{\sin\left[2(m-n)\pi\right]}{m-n} - \frac{1}{2}\frac{\sin\left[2(m+n)\pi\right]}{m+n} - \frac{1}{2}\frac{\sin 0}{m-n} + \frac{1}{2}\frac{\sin 0}{m+n} = 0$

Επειδή : $\sin 2k\pi = \sin 0 = 0, \forall$ ακέραιο k

8.4

$I_n = \int_{0}^{\pi/2} \sin^n x\, dx = -\int_{0}^{\pi/2} \sin^{n-1}x\, d\left(\cos x\right) = \underbrace{-\sin^{n-1}x \cos x\Big|_{0}^{\pi/2}}_{=0} + \int_{0}^{\pi/2}(n-1)\sin^{n-2}x \cos^2 x\, dx$

$= (n-1)\int_{0}^{\pi/2}\sin^{n-2}x\left(1-\sin^2 x\right)dx = (n-1)\int_{0}^{\pi/2}\sin^{n-2}x\, dx - (n-1)I_n$

$\Rightarrow I_n + (n-1)I_n = (n-1)I_{n-2} \Rightarrow I_n = \frac{n-1}{n}I_{n-2}$

8.4 (i) $\int \frac{\sin x}{\cos^2 x}dx = \int \tan x \sec x\, dx = \sec x + c$

(ii) $\int \sqrt{\cos x}\, \sin x\, dx = -\int \left(\cos x\right)^{1/2} d\left(\cos x\right) = -\frac{2}{3}\left(\cos x\right)^{3/2} + c$

(iii) $\int \tan^3 x \sec^5 x\, dx = \int \tan x \tan^2 x \sec^5 x\, dx = \int \tan x\left(\sec^2 x - 1\right)\sec^5 x\, dx$

$= \int \tan x \sec^7 x\, dx - \int \tan x \sec^5 x\, dx = \int (\tan x \sec x)\sec^6 x\, dx - \int (\tan x \sec x)\sec^4 x\, dx$

$= \int \sec^6 x\, d(\sec x) - \int \sec^4 x\, d(\sec x) = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + c$

8.5 (iv) $\int \text{cosec}^4 x\,dx = \int \text{cosec}^2 x\ \text{cosec}^2 x\,dx = \int \left(1+\cot^2 x\right)\ \text{cosec}^2 x\,dx$

$= \int \text{cosec}^2 x dx + \int \cot^2 x\ \text{cosec}^2 x dx = -\cot x - \frac{1}{3}\cot^3 x + c$

8.6 $A\sin(x+\varphi) = A\sin x\cos\varphi + A\sin\varphi\cos x = \sin x + \cos x$

$\Rightarrow \left.\begin{matrix} A\cos\varphi = 1 \\ A\sin\varphi = 1 \end{matrix}\right\} \Rightarrow \tan\varphi = 1 \Rightarrow \varphi = \pi/4 \Rightarrow A\cos\pi/4 = 1 \Rightarrow A = \sqrt{2}$

$\Rightarrow \int \frac{dx}{\sin x + \cos x} = \int \frac{dx}{\sqrt{2}\sin\left(x + \pi/4\right)} = \frac{1}{\sqrt{2}}\int \text{cosec}\left(x + \pi/4\right)dx$

$= -\frac{1}{\sqrt{2}}\ln\left|\text{cosec}\left(x+\pi/4\right) + \cot\left(x + \pi/4\right)\right| + c$

Με παρόμοιο τρόπο:

$A\sin(x+\varphi) = A\sin x\cos\varphi + A\sin\varphi\cos x = \alpha\sin x + b\cos x$

$\Rightarrow \left.\begin{matrix} A\cos\varphi = \alpha \\ A\sin\varphi = b \end{matrix}\right\} \Rightarrow \left.\begin{matrix} A^2\cos^2\varphi = \alpha^2 \\ A^2\sin^2\varphi = b^2 \end{matrix}\right\} \Rightarrow A = \sqrt{\alpha^2 + b^2},\ \tan\varphi = b/\alpha \Rightarrow \varphi = \tan^{-1}\left(b/\alpha\right)$

$\Rightarrow \int \frac{dx}{\alpha\sin x + b\cos x} = \int \frac{dx}{\sqrt{\alpha^2 + b^2}\sin(x+\varphi)}$

$= -\frac{1}{\sqrt{\alpha^2+b^2}}\ln\left|\text{cosec}\left(x + \tan^{-1}\left(b/\alpha\right)\right) + \cot\left(x + \tan^{-1}\left(b/\alpha\right)\right)\right| + c$

8.7 (i) $I = \int \frac{dx}{x^2\sqrt{x^2+25}}$

Θέτουμε: $x = 5\tan\theta \Rightarrow dx = 5\sec^2\theta\, d\theta$

$I = \int \frac{5\sec^2\theta}{25\tan^2\theta\sqrt{25\tan^2\theta + 25}}d\theta = \int \frac{\sec^2\theta}{5\tan^2\theta\ 5\sec\theta}d\theta = \frac{1}{25}\int \frac{\sec\theta}{\tan^2\theta}d\theta$

$= \frac{1}{25}\int \frac{1/\cos\theta}{\sin^2\theta/\cos^2\theta}d\theta = \frac{1}{25}\int \frac{\cos\theta}{\sin^2\theta}d\theta = \frac{1}{25}\int \cot\theta\,\text{cosec}\,\theta\, d\theta = -\frac{1}{25}\text{cosec}\ \theta$

Τώρα: $\frac{1}{\cos^2\theta} = \sec^2\theta = 1 + \tan^2\theta = 1 + 25x^2 \Rightarrow \cos^2\theta = \frac{1}{1+25x^2} \Rightarrow \sin^2\theta = \frac{25x^2}{1+25x^2}$

$\Rightarrow \text{cosec}\theta = \frac{1}{\sin\theta} = \frac{\sqrt{1+25x^2}}{5x}$

$\Rightarrow \int \frac{dx}{x^2\sqrt{x^2+25}} = \frac{1}{75}\frac{\sqrt{1+25x^2}}{x} + c$

8.7 (ii) $I=\int \frac{\cos x\, dx}{\sqrt{2-\sin^2 x}}$

Θέτουμε : $u=\sin x \Rightarrow du=\cos x\, dx$

$$\Rightarrow I=\int \frac{du}{\sqrt{2-u^2}}=\int \frac{1}{\sqrt{2}} \frac{du}{\sqrt{1-\left(u/\sqrt{2}\right)^2}}=\frac{1}{\cancel{\sqrt{2}}}\cancel{\sqrt{2}}\sin^{-1}\left(u/\sqrt{2}\right)=\sin^{-1}\left(\sin x/\sqrt{2}\right)+c$$

(iii) $I=\int_{\sqrt{2}}^{2} \frac{\sqrt{2x^2-4}}{x} dx$

Θέτουμε: $x=\sqrt{2}\sec\theta \Rightarrow dx=\sqrt{2}\sec\theta\tan\theta$

Επίσης: $x=2 \Rightarrow \sec\theta=\frac{2}{\sqrt{2}} \Rightarrow \cos\theta=\frac{\sqrt{2}}{2} \Rightarrow \theta=\frac{\pi}{4}$ και $x=\sqrt{2} \Rightarrow \sec\theta=1 \Rightarrow \theta=0$

$$I=\int_0^{\pi/4} \frac{2\tan\theta}{\sqrt{2}\sec\theta}\sqrt{2}\sec\theta\tan\theta\, d\theta=2\int_0^{\pi/4}\tan^2\theta\, d\theta=2\int_0^{\pi/4}\left(\sec^2\theta-1\right)d\theta$$

$$=2\left[\tan\theta-\theta\right]_0^{\pi/4}=2\tan\left(\pi/4\right)-2\frac{\pi}{4}-0=2-\frac{\pi}{2}$$

(iv) $I=\int_0^3 \frac{x^3}{\left(3+x^2\right)^{5/2}} dx$

Θέτουμε: $x=\sqrt{3}\tan\theta \Rightarrow dx=\sqrt{3}\sec^2\theta\, d\theta$

Επίσης: $x=3 \Rightarrow \tan\theta=3/\sqrt{3} \Rightarrow \theta=\frac{\pi}{3}$ και $x=0 \Rightarrow \theta=0$

$$I=\int_0^{\pi/3} \frac{3^{3/2}\tan^3\theta\,\sqrt{3}\sec^2\theta}{\left(3+3\tan^2\theta\right)^{5/2}}d\theta=\int_0^{\pi/3}\frac{3^2\tan^3\theta\,\sec^2\theta}{3^{5/2}\sec^5\theta}d\theta=\frac{1}{\sqrt{3}}\int_0^{\pi/3}\frac{\tan^3\theta}{\sec^3\theta}d\theta$$

$$=\frac{1}{\sqrt{3}}\int_0^{\pi/3}\sin^3\theta\, d\theta=\frac{1}{\sqrt{3}}\int_0^{\pi/3}\sin\theta\left(1-\cos^2\theta\right)d\theta=\frac{1}{\sqrt{3}}\int_0^{\pi/3}\left(\sin\theta-\sin\theta\cos^2\theta\right)d\theta$$

$$=\frac{1}{\sqrt{3}}\left[-\cos\theta+\frac{1}{3}\cos^3\theta\right]_0^{\pi/3}=\frac{5}{24\sqrt{3}}$$

8.8 $I=\int \frac{x}{x^2+4}dx$

<u>1ος τρόπος:</u>

Θέτω: $u=x^2+4 \Rightarrow du=2x\,dx$

$$I=\frac{1}{2}\int \frac{du}{u}=\frac{1}{2}\ln|u|+c=\frac{1}{2}\ln\left(x^2+4\right)+c$$

<u>2ος τρόπος:</u>

Θέτω: $x=2\tan\theta \Rightarrow dx=2\sec^2\theta$

$$I=\int \frac{2\tan\theta\, 2\sec^2\theta}{4\tan^2\theta+4}d\theta=\int \frac{\tan\theta \sec^2\theta}{\sec^2\theta}d\theta=\int \frac{\sin\theta}{\cos\theta}d\theta=-\int \frac{d(\cos\theta)}{\cos\theta}=-\ln|\cos\theta|+c_1$$

$$=-\ln\left|\frac{1}{\sec\theta}\right|+c_1=\ln|\sec\theta|+c_1=\ln\frac{\sqrt{x^2+4}}{2}+c_1=\frac{1}{2}\ln\left(x^2+4\right)\underbrace{-\ln 2+c_1}_{=c}=\frac{1}{2}\ln\left(x^2+4\right)+c$$

8.9 (i) $I=\int \frac{dx}{16x^2+16x+5}=\int \frac{dx}{(4x+2)^2+1}=\frac{1}{4}\tan^{-1}(4x+2)+c$

(ii) $I=\int \frac{e^x dx}{\sqrt{1+e^x+e^{2x}}}$

Θέτω: $u=e^x \Rightarrow du=e^x dx$

$$I=\int \frac{du}{\sqrt{u^2+u+1}}=\int \frac{du}{\sqrt{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}}=\int \frac{2}{\sqrt{3}}\frac{du}{\sqrt{\left(\frac{u+1/2}{\sqrt{3}/2}\right)^2+1}}=\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{2}\sinh^{-1}\left(\frac{2u+1}{\sqrt{3}}\right)$$

$$=\sinh^{-1}\left(\frac{2e^x+1}{\sqrt{3}}\right)+c$$

(iii) $I=\int \frac{\cos x\, dx}{\sin^2 x-6\sin x+12}$

Θέτουμε: $u=\sin x \Rightarrow du=\cos x\, dx$

$$I=\int \frac{du}{u^2-6u+12}=\int \frac{du}{(u-3)^2+3}=\frac{1}{3}\int \frac{du}{\left(\frac{u-3}{\sqrt{3}}\right)^2+1}=\frac{1}{3}\sqrt{3}\tan^{-1}\left(\frac{u-3}{\sqrt{3}}\right)+c=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sin x-3}{\sqrt{3}}\right)+c$$

(iv) $\int \frac{2x+3}{4x^2+4x+5}dx=\frac{1}{4}\int \frac{8x+12}{4x^2+4x+5}dx=\frac{1}{4}\int \frac{8x+4}{4x^2+4x+5}dx+\frac{1}{4}\int \frac{8}{(2x+1)^2+4}dx=$

$$=\frac{1}{4}\ln\left(4x^2+4x+5\right)+2\int \frac{1}{4}\frac{dx}{\left(\frac{2x+1}{2}\right)^2+1}=\frac{1}{4}\ln\left(4x^2+4x+5\right)+\frac{1}{2}\tan^{-1}\left(x+\tfrac{1}{2}\right)+c$$

8.9 (v) $I=\int \frac{dx}{x^3+x}$

$$\frac{1}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+\Gamma}{x^2+1}\Rightarrow \ldots A=1, B=-1, \Gamma=0$$

$$I=\int\frac{dx}{x}-\int\frac{xdx}{x^2+1}=\ln x-\frac{1}{2}\ln\left(x^2+1\right)+c$$

(vi) $\int\frac{x^4+6x^3+10x^2+x}{x^2+6x+10}dx=\int x^2dx+\int\frac{x}{x^2+6x+10}dx=\frac{x^3}{3}+\frac{1}{2}\int\frac{2x+6-6}{x^2+6x+10}dx$

$$=\frac{x^3}{3}+\frac{1}{2}\int\frac{2x+6}{x^2+6x+10}dx-\frac{1}{2}\int\frac{6dx}{(x+3)^2+1}=\frac{x^3}{3}+\frac{1}{2}\ln\left(x^2+6x+10\right)-3\tan^{-1}(x+3)+c$$

(vii) $\int\frac{dx}{1+e^x}=\int\frac{e^{-x}}{e^{-x}+1}dx=-\ln\left(e^{-x}+1\right)$

(viii) $I=\int\frac{\sec^2xdx}{\tan^3x-\tan^2x}$

Θέτουμε: $u=\tan x\Rightarrow du=\sec^2xdx$

$$I=\int\frac{du}{u^3-u^2}=\int\left[-\frac{1}{u}-\frac{1}{u^2}+\frac{1}{u-1}\right]du=-\ln u+\frac{1}{u}+\ln(u-1)+c=\frac{1}{\tan x}-\ln\left|\tan x\right|+\ln\left|\tan x-1\right|+c$$

8.10 $x^4+1=\left(x^2+\alpha x+1\right)\left(x^2+bx+1\right)\Rightarrow\ldots\alpha=\sqrt{2}, b=-\sqrt{2}$

$$I=\int_0^1\frac{x}{x^4+1}dx$$

$$\frac{x}{x^2+1}=\frac{Ax+B}{x^2+\sqrt{2}x+1}+\frac{\Gamma x+\Delta}{x^2-\sqrt{2}x+1}\ldots\Rightarrow A=\Gamma=0, B=-\frac{\sqrt{2}}{4}, \Delta=\frac{\sqrt{2}}{4}$$

$$I=-\frac{\sqrt{2}}{4}\int_0^1\frac{dx}{x^2+\sqrt{2}x+1}+\frac{\sqrt{2}}{4}\int_0^1\frac{dx}{x^2-\sqrt{2}x+1}=I_1+I_2$$

$$I_1=-\frac{\sqrt{2}}{4}\int_0^1\frac{dx}{x^2+\sqrt{2}x+1}=-\frac{\sqrt{2}}{4}\int_0^1\frac{dx}{\left(x+\sqrt{2}/2\right)^2+1/2}=-\frac{\sqrt{2}}{4}\int_0^1 2\frac{dx}{\left(\sqrt{2}\left(x+\sqrt{2}/2\right)\right)^2+1}$$

$$=-\frac{\sqrt{2}}{2}\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\left(x+\sqrt{2}/2\right)\right)\Big|_0^1=-\frac{1}{2}\tan^{-1}\left(\sqrt{2}+1\right)+\frac{\pi}{8}$$

$$I_2=\frac{\sqrt{2}}{4}\int_0^1\frac{dx}{x^2-\sqrt{2}x+1}=\frac{\sqrt{2}}{4}\int_0^1\frac{dx}{\left(x-\sqrt{2}/2\right)^2+1/2}=\frac{\sqrt{2}}{4}\int_0^1 2\frac{dx}{\left(\sqrt{2}\left(x-\sqrt{2}/2\right)\right)^2+1}$$

$$=\frac{\sqrt{2}}{2}\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\left(x-\sqrt{2}/2\right)\right)\Big|_0^1=\frac{1}{2}\tan^{-1}\left(\sqrt{2}-1\right)+\frac{\pi}{8}$$

$$I=\frac{\pi}{8}-\frac{1}{2}\tan^{-1}\left(\sqrt{2}+1\right)+\frac{\pi}{8}+\frac{1}{2}\tan^{-1}\left(\sqrt{2}-1\right)=\frac{\pi}{4}-\frac{1}{2}\left[\tan^{-1}\left(\sqrt{2}+1\right)+\tan^{-1}\left(1-\sqrt{2}\right)\right]$$

$$=\frac{\pi}{4}-\frac{1}{2}\tan^{-1}\left[\frac{1+\sqrt{2}+1-\sqrt{2}}{1-\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)}\right]=\frac{\pi}{4}-\frac{1}{2}\tan^{-1}(1)=\frac{\pi}{4}-\frac{\pi}{8}=\frac{\pi}{8}$$

8.11 (i) $I=\int \frac{1+\sqrt{x}}{1-\sqrt{x}}dx$

Θέτουμε : $u=\sqrt{x} \Rightarrow du=\frac{1}{2}\frac{1}{\sqrt{x}}dx \Rightarrow 2udu=dx$

$$I=\int \frac{1+u}{1-u}2udu=2\int \frac{u^2+u}{1-u}du=2\int\left[-u+2\frac{u}{1-u}\right]du=-2\int udu-4\int \frac{-u+1-1}{1-u}du$$

$$=-u^2-4\int 1du+4\int \frac{1}{1-u}du=-u^2-4u-4\ln(u-1)+c=-\sqrt{x}-4\sqrt{x}-4\ln(\sqrt{x}-1)+c$$

(ii) $I=\int e^{\sqrt{x}}dx$

Θέτουμε : $u=\sqrt{x} \Rightarrow 2udu=dx$

$$I=\int 2ue^u du=2ue^u+\int 2e^u du=2ue^u+2e^u+c=2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+c$$

(iii) $I=\int \frac{dx}{2+\sin x}$

Θέτουμε : $t=\tan\left(\frac{x}{2}\right) \Rightarrow dx=\frac{2dt}{1+t^2}, \sin x=\frac{2t}{1+t^2}$

$$I=\int \frac{\frac{2dt}{1+t^2}}{2+\frac{2t}{1+t^2}}=\int \frac{2dt}{2t^2+2t+2}=\int \frac{dt}{(t+1/2)^2+3/4}=\int \frac{4}{3}\frac{dt}{\left(\frac{t+1/2}{\sqrt{3}/2}\right)^2+1}=\frac{4}{3}\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)+c$$

$$=\frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{2\tan\left(x/2\right)+1}{\sqrt{3}}\right)+c$$

(iv) $I=\int \frac{\cos x}{2-\cos x}dx$

Θέτουμε : $t=\tan\left(\frac{x}{2}\right) \Rightarrow dx=\frac{2dt}{1+t^2}, \cos x=\frac{1-t^2}{1+t^2}$

$$I=\int \frac{\frac{1-t^2}{1+t^2}}{2-\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt=2\int \frac{1-t^2}{\left(1+3t^2\right)\left(1+t^2\right)}dt$$

$$\frac{1-t^2}{\left(1+3t^2\right)\left(1+t^2\right)}=\frac{At+B}{1+t^2}+\frac{\Gamma t+\Delta}{3t^2+1} \Rightarrow (At+B)\left(3t^2+1\right)+\left(\Gamma t+\Delta\right)\left(1+t^2\right)=1-t^2$$

$$3At^3+At+3Bt^2+B+\Gamma t+\Gamma t^2+\Delta+\Delta t^2=1-t^2 \Rightarrow A=0, B=-1, \Gamma=0, \Delta=2$$

$$\Rightarrow I=2\int\left(\frac{-1}{1+t^2}+\frac{2}{3t^2+1}\right)dt=-2\int\frac{dt}{1+t^2}+4\int\frac{dt}{1+\left(\sqrt{3}t\right)^2}=-2\tan^{-1}t+\frac{4}{\sqrt{3}}\tan^{-1}\left(t\sqrt{3}\right)+c$$

$$=-2\tan^{-1}\left(\tan x/2\right)+\frac{4}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x/2\right)+c$$

8.12 $x=\frac{1}{u} \Rightarrow dx=-\frac{1}{u^2}$

(i) $\int \frac{\sqrt{4-x^2}}{x^4} = \int \frac{\sqrt{4-1/u^2}}{1/u^{\cancel{4}2}}\left(-\frac{1}{\cancel{u^2}}\right)du = -\int u^2\sqrt{\frac{4u^2-1}{u^2}}du = -\frac{1}{8}\int 8u\left(4u^2-1\right)^{1/2}du$

$= -\frac{1}{8}\frac{2}{3}\left(4u^2-1\right)^{3/2}+c = -\frac{1}{12}\left(4\frac{1}{x^2}-1\right)^{3/2}+c = -\frac{\left(4-x^2\right)^{3/2}}{12x^3}+c$

(ii) $\int \frac{dx}{x^2\sqrt{3-x^2}} = \int \frac{-1/\cancel{u^2}}{1/\cancel{u^2}\sqrt{3-1/u^2}}du = -\int \frac{u}{\sqrt{3u^2-1}}du = -\frac{1}{6}\int \frac{6u}{\left(3u^2-1\right)^{1/2}}du$

$= -\frac{1}{6}2\left(3u^2-1\right)^{1/2}+c = -\frac{1}{3}\left(\frac{3}{x^2}-1\right)^{1/2}+c = -\frac{\left(3-x^2\right)^{1/2}}{3x}+c$

(iii) $\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{-1/\cancel{u^2}}{1/\cancel{u^2}\sqrt{1/u^2+1}}du = -\frac{1}{2}\int \frac{2u}{\sqrt{1+u^2}}du = -\frac{1}{2}2\sqrt{1+u^2}+c = -\sqrt{1+1/x^2}+c$

(iv) $\int \frac{\sqrt{x^2-5}}{x^4}dx = \int \frac{\sqrt{1/u^2-5}}{1/u^{\cancel{4}2}}\left(-\frac{1}{\cancel{u^2}}\right)du = -\int \frac{u^{\cancel{2}}\sqrt{1-5u^2}}{\cancel{u}}du = \frac{1}{10}\int(-10)u\sqrt{1-5u^2}du$

$= \frac{1}{10}\frac{2}{3}\left(1-5u^2\right)^{3/2}+c = \frac{1}{15}\left(1-\frac{5}{x^2}\right)^{3/2}+c$

8.13 (i) $\int \frac{1+x}{\sqrt{x}}dx = \int\left(\frac{1}{x^{1/2}}+x^{1/2}\right)dx = 2x^{1/2}+\frac{2}{3}x^{3/2}+c$

(ii) $I=\int \frac{\sqrt{1-x^2}}{x^2}dx$

Θέτουμε: $x=\sin\theta \Rightarrow dx=\cos\theta\, d\theta$

$\Rightarrow I=\int \frac{\sqrt{1-\sin^2\theta}}{\sin^2\theta}\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}d\theta = \int \cot^2\theta\, d\theta = \int\left(\operatorname{cosec}^2\theta-1\right)d\theta$

$= -\cot\theta-\theta+c = -\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}-\theta+c = -\frac{\sqrt{1-x^2}}{x}-\sin^{-1}x+c$

(iii) $I=\int \frac{dx}{x^{1/2}+x^{1/4}}$

$u=x^{1/4} \Rightarrow x=u^4 \Rightarrow dx=4u^3du$

$\Rightarrow \int \frac{4u^{\cancel{3}2}du}{\cancel{u}(u+1)} = 4\int\left[u-1+\frac{1}{u+1}\right]du = 4\frac{u^2}{2}-4u+4\ln(u+1)+c = 2\sqrt{x}-4x^{1/4}+\ln\left(x^{1/4}+1\right)+c$

8.13 (iv) $I=\int \frac{dx}{\sin x-\tan x}=\int \frac{dx}{\sin x-\sin x/\cos x}=\int \frac{\cos x}{\sin x\cos x-\sin x}dx=\int \frac{\cos x}{\sin x(\cos x-1)}dx$

Θέτουμε : $t=\cos x \Rightarrow dt=-\sin x dx \Rightarrow dx=-\frac{dt}{\sin x}=-\frac{dt}{\sqrt{1-t^2}}$

$\Rightarrow I=\int \frac{t}{\sqrt{1-t^2}(t-1)}\left(-\frac{1}{\sqrt{1-t^2}}\right)dt=-\int \frac{t}{(1-t^2)(t-1)}dt=-\int \frac{t}{(1-t)(1+t)(t-1)}dt=\int \frac{t}{(t-1)^2(t+1)}dt$

$\frac{t}{(t-1)^2(t+1)}=\frac{A}{t+1}+\frac{B}{t-1}+\frac{\Gamma}{(t-1)^2}\ldots \Rightarrow A=-1/4,\ B=1/4,\ \Gamma=1/2$

$\Rightarrow I=-\frac{1}{4}\int \frac{dt}{t+1}+\frac{1}{4}\int \frac{dt}{t-1}+\frac{1}{2}\int \frac{dt}{(t-1)^2}=-\frac{1}{4}\ln|t+1|+\frac{1}{4}\ln|t-1|-\frac{1}{2(t-1)}+c$

$=-\frac{1}{4}\ln(\cos x+1)+\frac{1}{4}\ln(\cos x-1)-\frac{1}{2(\cos x-1)}+c$

8.14 $x=\pi-u \Rightarrow dx=-du$, $x=\pi \rightarrow u=0$ και $x=0 \rightarrow u=\pi$

$\int_0^\pi x f(\sin x)dx=-\int_\pi^0 (\pi-u)f\left(\sin(\pi-u)\right)du=\int_0^\pi (\pi-u)f(\sin u)du=\pi\int_0^\pi f(\sin u)du-\int_0^\pi uf(\sin u)du$

$\Rightarrow 2\int_0^\pi x f(\sin x)dx=\pi\int_0^\pi f(\sin x)dx \Rightarrow \int_0^\pi xf(\sin x)dx=\frac{\pi}{2}\int_0^\pi f(\sin x)dx$

8.15 $\int_0^\pi \frac{x\sin x}{2-\sin^2 x}dx=\frac{\pi}{2}\int_0^\pi \frac{\sin x}{2-\sin^2 x}dx=\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx$

$=\frac{\pi}{2}\left[\tan^{-1}(\cos x)\right]_0^\pi=-\frac{\pi^2}{2}$

8.16 $I=\int \frac{\sqrt{x+1}}{(x-1)^{5/2}}dx$

Θέτω $x-1=\frac{1}{u} \Rightarrow dx=-\frac{1}{u^2}du$

$I=\int \frac{\sqrt{1/u+2}}{1/u^{5/2}}\left(-\frac{1}{u^2}\right)du=-\int \frac{\sqrt{1+2u}}{u^{1/2}}u^{1/2}du=-\frac{1}{2}\int 2\sqrt{1+2u}du=-\frac{2}{3}\frac{1}{2}\left(2u+1\right)^{3/2}+c$

$=-\frac{1}{3}\left(2u+1\right)^{3/2}+c=-\frac{1}{3}\left(2\frac{1}{x-1}+1\right)^{3/2}+c$

8.17 Θέτουμε $x=\alpha-u \Rightarrow dx=-du$, $x=\alpha \to u=0$, $x=0 \to u=\alpha$

$$\int_0^{\alpha} f(x)dx=\int_{\alpha}^{0} f(\alpha-u))(-du)=\int_0^{\alpha} f(\alpha-x)dx$$

(i) Για $\alpha=\pi$ έχουμε

$$\int_0^{\pi}\frac{x\,\sin x}{1+\cos^2 x}dx=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx=\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx-\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow 2\int_0^{\pi}\frac{x\,\sin x}{1+\cos^2 x}dx=\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow \int_0^{\pi}\frac{x\,\sin x}{1+\cos^2 x}dx=-\frac{\pi}{2}\int_0^{\pi}\frac{d(\cos x)}{1+\cos^2 x}dx=-\frac{\pi}{2}\Big[\tan^{-1}(\cos x)\Big]_0^{\pi}=-\frac{\pi}{2}\Big[\tan^{-1}(-1)-\tan^{-1}(1)\Big]$$

$$=-\frac{\pi}{2}\left(-\frac{\pi}{4}-\frac{\pi}{4}\right)=\frac{\pi^2}{4}$$

(ii) $$I=\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx=-\int_{\pi/2}^{0}\frac{\sin^2(\pi/2-u)}{\sin(\pi/2-u)+\cos(\pi/2-u)}du=\int_0^{\pi/2}\frac{\cos^2 x}{\sin x+\cos x}dx$$

$$=\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}-\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx$$

$$\Rightarrow 2\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx=\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}$$

$$\Rightarrow \int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx=\frac{1}{2}\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}$$ από άσκηση 8.6 έχουμε:

$$=\frac{1}{2}\left[-\frac{1}{\sqrt{2}}\ln\left|\operatorname{cosec}\left(\pi/4+x\right)+\cot\left(\pi/4+x\right)\right|\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\left\{\left[-\frac{1}{\sqrt{2}}\ln\left|\operatorname{cosec}\left(\pi/4+\pi/2\right)+\cot\left(\pi/4+\pi/2\right)\right|\right]-\left[-\frac{1}{\sqrt{2}}\ln\left|\operatorname{cosec}\left(\pi/4\right)+\cot\left(\pi/4\right)\right|\right]\right\}$$

$$=-\frac{1}{2\sqrt{2}}\ln\left|\frac{2}{\sqrt{2}}-1\right|+\frac{1}{2\sqrt{2}}\ln\left|\frac{2}{\sqrt{2}}+1\right|=\frac{1}{2\sqrt{2}}\ln\frac{2/\sqrt{2}+1}{2/\sqrt{2}-1}=\frac{1}{2\sqrt{2}}\ln\frac{2+\sqrt{2}}{2-\sqrt{2}}=\frac{1}{2\sqrt{2}}\ln\left(3+2\sqrt{2}\right)$$

$$=\frac{1}{\sqrt{2}}\ln\sqrt{3+2\sqrt{2}}=\frac{1}{\sqrt{2}}\ln\sqrt{3+2\sqrt{2}}=\frac{1}{\sqrt{2}}\ln\left(\sqrt{2}+1\right)$$

8.18 (i) $\int_1^2 (x-1)^2 \ln x\,dx = \frac{(x-1)^3}{3}\ln x\Big|_1^2 - \frac{1}{3}\int_1^2 \frac{(x-1)^3}{x}dx = \frac{1}{3}\ln 2 - \frac{1}{3}\int_1^2 \frac{x^3-3x^2+3x-1}{x}dx$

$\Rightarrow \frac{1}{3}\ln 2 - \frac{1}{3}\int_1^2\left(x^2-3x+3-\frac{1}{x}\right)dx = \frac{1}{3}\ln 2 - \frac{1}{3}\left[\frac{x^3}{3}-\frac{3x^2}{2}+3x-\ln x\right]_1^2 = \ldots = \frac{2}{3}\ln 2 - \frac{5}{18}$

(ii) $I = \int_0^{\pi/2}\frac{dx}{(\sin x+\cos x)^2} = \int_0^{\pi/2}\frac{dx}{\cos^2 x(\tan x+1)^2} = \int_0^{\pi/2}\frac{\sec^2 x\,dx}{(\tan x+1)^2}$

$= -\frac{1}{1+\tan x}\Big|_0^{\pi/2} = 1$

(iii) $I = \int_0^1\sqrt{\frac{1+x}{1-x}}dx$

Θέτουμε : $x=\sin t \Rightarrow dx=\cos t\,dt$, $x=1 \to t=\frac{\pi}{2}$, $x=0 \to t=0$

$I = \int_0^{\pi/2}\sqrt{\frac{1+\sin t}{1-\sin t}}\cos t\,dt = \int_0^{\pi/2}\sqrt{\frac{(1+\sin t)^2}{1-\sin^2 t}}\cos t\,dt = \int_0^{\pi/2}\frac{1+\sin t}{\cos t}\cos t\,dt = \int_0^{\pi/2}(1+\sin t)\,dt$

$= t-\cos t\Big|_0^{\pi/2} = \frac{\pi}{2}+1$

(iv) $I = \int_1^2\frac{dx}{(4-x)\sqrt{x-1}}$

Θέτουμε : $x=1+u^2 \Rightarrow dx=2udu$, $x=2 \to u=$

$I = \int\frac{2udu}{(3-u^2)\sqrt{u^2}} = \int\frac{2udu}{(3-u^2)u} = 2\int\frac{du}{\left(\sqrt{3}-u\right)\left(\sqrt{3}+u\right)} = 2\int\frac{1}{2\sqrt{3}}\frac{1}{\left(\sqrt{3}-u\right)} + 2\int\frac{1}{2\sqrt{3}}\frac{1}{\left(\sqrt{3}+u\right)}$

$= -\frac{1}{\sqrt{3}}\ln\left(\sqrt{3}-u\right) + \frac{1}{\sqrt{3}}\ln\left(\sqrt{3}+u\right) \Rightarrow I = -\frac{1}{\sqrt{3}}\ln\left(\sqrt{3}-\sqrt{x-1}\right) + \frac{1}{\sqrt{3}}\ln\left(\sqrt{3}+\sqrt{x-1}\right)\Big|_1^2$

$= -\frac{1}{\sqrt{3}}\ln\left(\sqrt{3}-1\right) + \frac{1}{\sqrt{3}}\ln\left(\sqrt{3}+1\right) + \frac{1}{\sqrt{3}}\ln\sqrt{3} - \frac{1}{\sqrt{3}}\ln\sqrt{3} = \frac{1}{\sqrt{3}}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$

8.19 $I_n = \int_0^{\pi/2} e^{\alpha x}\cos^n x\,dx = \frac{1}{\alpha}e^{\alpha x}\cos^n x\Big|_0^{\pi/2} - \frac{1}{\alpha}\int_0^{\pi/2} n e^{\alpha x}\cos^{n-1}x(-\sin x)dx$

$= -\frac{1}{\alpha} + \frac{n}{\alpha^2}\left[e^{\alpha x}\cos^{n-1}x\sin x\right]\Big|_0^{\pi/2} - \frac{n}{\alpha^2}\int_0^{\pi/2}\left[e^{\alpha x}(n-1)\cos^{n-2}x(-1)\sin^2 x+\cos^{n-1}x\cos x\right]dx$

$= -\frac{1}{\alpha} - \frac{n}{\alpha^2}\int_0^{\pi/2} e^{\alpha x}\left(-(n-1)\cos^{n-2}x(1-\cos^2)x+\cos^n x\right)dx = -\frac{1}{\alpha} - \frac{n}{\alpha^2}\left[-(n-1)I_{n-2}+nI_n\right]$

$\Rightarrow I_n = \frac{n(n-1)}{n^2+\alpha^2}I_{n-2} - \frac{\alpha}{n^2+\alpha^2}$

8.20 (i) $I=\int \frac{e^x}{2+2e^x+e^{2x}}dx$

Θέτουμε : $u=e^x \Rightarrow du=e^x dx$

$$I=\int \frac{du}{u^2+2u+2}=\int \frac{du}{(u+1)^2+1}=\tan^{-1}(u+1)+c=\tan^{-1}(e^x+1)+c$$

(ii) $I=\int \frac{\tan^{-1}(x+2)}{x^2+4x+5}dx=\int \frac{\tan^{-1}(x+2)}{(x+2)^2+1}dx$

Θέτουμε : $u=x+2 \Rightarrow du=dx$

$$I=\int \frac{\tan^{-1}u}{u^2+1}du=\frac{1}{2}\left(\tan^{-1}u\right)^2+c=\frac{1}{2}\left(\tan^{-1}(x+2)\right)^2+c$$ -