ΛΥΣΕΙΣ ΑΣΚΗΣΕΩΝ
ΚΕΦΑΛΑΙΟ 7^{o}

7.1 $(\sinh x+\cosh x)^n = \sinh nx + \cosh nx$

$$(\sinh x+\cosh x)^n = \left[\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}\right]^n = \left[e^x\right]^n$$

$$= e^{nx} = \frac{e^{nx} - e^{-nx}}{2} + \frac{e^{nx} + e^{-nx}}{2} = \sinh nx + \cosh nx$$

7.2 (i) $|\tanh x| = \frac{|\sinh x|}{|\cosh x|} = \frac{|\sinh x|}{\sqrt{1+\sinh^2 x}} < \frac{|\sinh x|}{\sqrt{\sinh^2 x}},\ x \neq 0$

$$\Rightarrow |\tanh x| < \frac{|\sinh x|}{|\sinh x| = 1},\ \forall x \neq 0$$

και επειδή $\tanh 0 = \frac{e^0 - e^0}{e^0 + e^0} = 0 \Rightarrow |\tanh x| < 1,\ \forall x \in \mathbb{R}$

(ii) $\lim\limits_{x\to+\infty} \tanh x = \lim\limits_{x\to+\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x\to+\infty} \frac{1-e^{-2x}}{1+e^{-2x}} = 1$

(iii) $\lim\limits_{x\to-\infty} \tanh x = \lim\limits_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x\to-\infty} \frac{e^{2x}-1}{e^{2x}+1} = -1$

7.3 (i) $y = \left(\tanh^{-1}x\right)^2 \Rightarrow y' = \frac{2\tanh^{-1}x}{1-x^2}$

(ii) $y = \ln\left(\cosh^{-1}x\right) \Rightarrow y' = \frac{1}{\cosh^{-1}x\sqrt{x^2-1}}$

(iii) $y = \cosh^{-1}\left(\sinh^{-1}x\right) \Rightarrow y' = \frac{1}{\sqrt{\left(1+x^2\right)\left(\left[\sinh^{-1}x\right]^2 - 1\right)}}$

7.4 $\frac{d}{dx}\left[\operatorname{sech}^{-1}|x|\right] = \frac{d}{dx}\left[\operatorname{sech}^{-1}\left(\sqrt{x^2}\right)\right] = \frac{-1}{\sqrt{x^2}\sqrt{1-x^2}}\,\frac{2x}{2\sqrt{x^2}} = \frac{-1}{x\sqrt{1-x^2}}$

7.5 $u = \tanh^{-1}\left(-\frac{3}{5}\right) \Leftrightarrow \tanh u = -\frac{3}{5}$

(i) $\tanh^2 u = 1 - \operatorname{sech}^2 u \Rightarrow \operatorname{sech} u = \sqrt{1-\tanh^2 u}$

Άρα : $\cosh u = \frac{1}{\sqrt{1-9/25}} = \frac{1}{\sqrt{16/25}} = \frac{5}{4}$

(ii) $\tanh u = \frac{\sinh u}{\cosh u} \Rightarrow \sinh u = \cosh u \cdot \tanh u = -\frac{3}{4}$

(iii) $\cosh 2u = \cosh^2 u + \sinh^2 u = \frac{25}{16} + \frac{9}{16} = \frac{34}{16}$

7.6 $u=\text{cosech}^{-1}\left(-\frac{5}{12}\right) \Rightarrow \text{cosech } u=-\frac{5}{12}$

(i) $\coth^2 u=1+\text{cosech}^2 u \Rightarrow \coth^2 u=\frac{169}{144}$

(ii) $\sinh u=\frac{1}{\text{cosech } u}=-\frac{12}{5}$

(iii) $\cosh=\sqrt{1+\sinh^2 u}=\sqrt{1+\left(-\frac{12}{5}\right)}=\frac{13}{5}$

(iv) $\sinh 2u=2\sinh u \cosh u=\left(-\frac{12}{5}\right)\left(\frac{13}{5}\right)=-\frac{312}{25}$

7.7 $f(x)=\tanh^{-1} x \Rightarrow f'(x)=\frac{1}{1-x^2}$

Η f είναι συνεχής στο διάστημα $[0,x,]$, $x<1$ και παραγωγίσιμη σε αυτό

Άρα από το θεώρημα Μέσης Τιμής: $\exists c \in (0,x): f'(c)=\frac{f(x)-f(0)}{x-0}$

Δηλαδή : $\frac{1}{1-c^2}=\frac{\tanh^{-1}x-\cancelto{0}{\tanh^{-1}0}}{x}=\frac{\tanh^{-1}x}{x}, \forall 0<c<x$

Όμως : $0<c^2<x^2 \Rightarrow -x^2<-c^2<0 \Rightarrow 1-x^2<1-c^2<1 \Rightarrow 1<\frac{1}{1-c^2}<\frac{1}{1-x^2}$

$\Rightarrow 1<\frac{\tanh^1 x}{x}<\frac{1}{1-x^2} \Rightarrow x<\tanh^1 x<\frac{x}{1-x^2}$

7.8 $\cosh^{-1}\left(\frac{5}{4}\right)+\cosh^{-1}\left(\frac{5}{3}\right)=\ln x-\ln(x-1),\ x>1$

$\Leftrightarrow \ln\left(\frac{5}{4}+\sqrt{\frac{25}{16}-1}\right)+\ln\left(\frac{5}{3}+\sqrt{\frac{25}{9}-1}\right)=\ln\frac{x}{x-1} \Leftrightarrow \ln\left(\frac{5}{4}+\frac{3}{4}\right)+\ln\left(\frac{5}{3}+\frac{4}{3}\right)=\ln\frac{x}{x-1}$

$\ln 2+\ln 3=\ln\frac{x}{x-1} \Leftrightarrow \ln 6=\ln\frac{x}{x-1} \Leftrightarrow 6=\frac{x}{x-1} \Leftrightarrow 6(x-1)=x \Rightarrow x=\frac{6}{5}$

7.9 $12\cosh^2 x-25\sinh x=0 \Leftrightarrow 12\left(1+\sinh^2 x\right)-25\sinh x=0$

$\Leftrightarrow 12+12\sinh^2 x-25\sinh x=0 \Leftrightarrow \sinh x=\frac{25\pm\sqrt{25^2-4(12)(12)}}{2(12)}=\frac{25\pm 7}{24}$

$\Rightarrow \sinh x=\frac{4}{3} \Rightarrow x=\ln\left(\frac{4}{3}+\sqrt{1+\frac{16}{9}}\right) \Rightarrow x=\ln 3$

και $\sinh x=\frac{3}{4} \Rightarrow x=\ln\left(\frac{3}{4}+\sqrt{1+\frac{9}{16}}\right) \Rightarrow x=\ln 2$

7.10 $5\cosh x-3\sinh x=R\cosh(x-\alpha)\Leftrightarrow 5\cosh x-3\sinh x=\ R\cosh x\cosh\alpha-R\sinh x\sinh\ \alpha$

$$\Leftrightarrow \begin{matrix} R\cosh\alpha=5 \\ R\sinh\alpha=3 \end{matrix} \Leftrightarrow \begin{matrix} R^2\cosh^2\alpha=25 \\ R^2\sinh^2\alpha=9 \end{matrix} \Leftrightarrow R^2\cosh^2\alpha-R^2\sinh^2\alpha=25-9=16\Rightarrow R^2=16\Rightarrow R=4$$

Επίσης: $4\cosh\alpha=5\Rightarrow \cosh\alpha=\frac{5}{4}\Rightarrow \alpha=\cosh^{-1}\left(\frac{5}{4}\right)=\ln\left(\frac{5}{4}+\sqrt{\frac{25}{16}-1}\right)=\ln\left(\frac{5}{4}+\frac{3}{4}\right)=\ln2$

Τώρα η εξίσωση γίνεται: $5\cosh x-3\sinh x=4\cosh(x-\ln2)$

$$5\cosh x-3\sinh x=5\sinh\left(x-\ln2\right)=4\cosh(x-\ln2)\Rightarrow \tanh(x-\ln2)=\frac{4}{5}\Rightarrow x-\ln2=\tanh^{-1}\left(\frac{4}{5}\right)$$

$$\Rightarrow x=\ln2+\frac{1}{2}\ln\left(\frac{1+4/5}{1\ 4/5}\right)=\ln2+\frac{1}{2}\ln9=\ln2+\ln3=\ln6$$