ΛΥΣΕΙΣ ΑΣΚΗΣΕΩΝ
ΚΕΦΑΛΑΙΟ 6^{o}

6.1 (i) $\int \sqrt{x}\left(x^2+4x^3\right)dx=\int x^{5/2}+4x^{7/2}dx=\frac{2}{7}x^{7/2}+\frac{2}{9}x^{9/2}+c$

(ii) $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx=\frac{2}{3}x^{3/2}-2x^{1/2}+c$

(iii) $\int\frac{\sin x}{\cos^2 x}dx=\int \sec x\ \tan x\ dx=\sec x+c$

(iv) $\int_{\pi/6}^{\pi/2}\left(x+\frac{2}{\sin^2 x}\right)dx=\int_{\pi/6}^{\pi/2}\left(x+2\operatorname{cosec}^2 x\right)dx=\frac{1}{2}x^2-2\cot x\Big|_{\pi/6}^{\pi/2}=\frac{\pi^2}{9}+2\sqrt{3}$

(v) $\int_{\pi/4}^{\pi/2}\frac{\cos x}{\sin^2 x}dx=\int_{\pi/4}^{\pi/2}\operatorname{cosec}x\ \cot x dx=-\operatorname{cosec}x\Big|_{\pi/4}^{\pi/2}=\sqrt{2}-1$

(vi) $\int_{-2}^{2}\left|2x-5\right|dx=\int_{-2}^{2}(5-2x)dx=5x-x^2\Big|_{-2}^{2}=20$

6.2 $f'(x)=6-5\sin x \Rightarrow f(x)=\int(6-5\sin x)\ dx \Rightarrow f(x)=6x+\frac{5}{2}\sin x+c$

$f(0)=3 \Rightarrow 3=\frac{5}{2}\sin x+c \Rightarrow c=\frac{1}{2}$

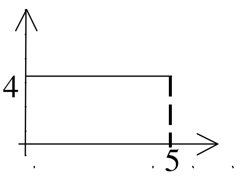
Άρα $f(x)=6x+\frac{5}{2}\sin x+\frac{1}{2}$

6.3 $f'(x)=\sqrt{x} \Rightarrow f(x)=\int\sqrt{x}\ dx \Rightarrow f(x)=\frac{2}{3}x^{3/2}+c$

$f(1)=5 \Rightarrow 5=\frac{2}{3}+c \Rightarrow c=\frac{13}{3}$

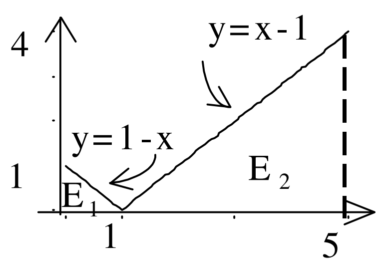
Άρα $f(x)=\frac{2}{3}x^{3/2}+\frac{13}{3}$

6.4 (i)



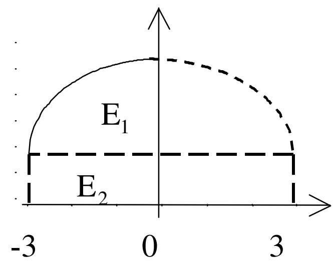
$E=4\times5=20$

(ii)



$E=E_1+E_2=\frac{1}{2}1\times1+\frac{1}{2}4\times4=\frac{1}{2}+8=\frac{17}{2}$

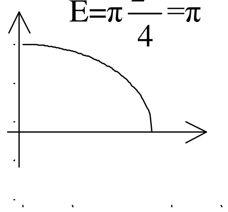
6.4 (iii) $y=2+\sqrt{9-x^2} \Rightarrow (y-2)^2+x^2=3^2$



$$E=E_1+E_2=\pi\frac{3^2}{4}+3\times 2 \Rightarrow E=\frac{9}{4}\pi+6$$

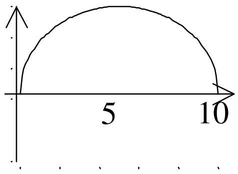
(iv) $y=\sqrt{4-x^2} \Rightarrow y^2+x^2=2^2$

$$E=\pi\frac{2^2}{4}=\pi$$



(v) $y=\sqrt{10x-x^2} \Rightarrow y^2+x^2-10x+25=25$

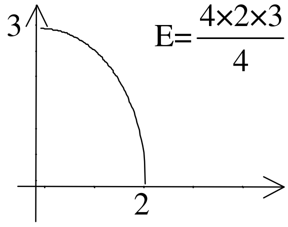
$$\Rightarrow y^2+(x-5)^2=5^2$$



$$E=\pi\frac{5^2}{2}=\pi\frac{25}{2}$$

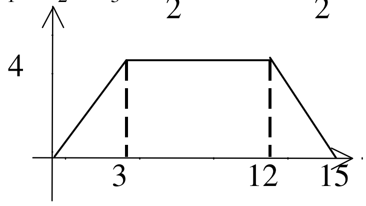
(vi) $y=\sqrt{9-\frac{9}{4}x^2} \Rightarrow \frac{y^2}{9}+\frac{x^2}{4}=1$

$$E=\frac{4\times 2\times 3}{4}=6$$



(vii) <u>$1^{ος}$ τρόπος:</u>

$$E=E_1+E_2+E_3=\frac{3\times 4}{2}+9\times 4+\frac{3\times 4}{2}=48$$



<u>$2^{ος}$ τρόπος:</u>

$$E=\frac{(\beta_1+\beta_2)^{\nu}}{2}=\frac{(15+9)^4}{2}=48$$

6.5 $\frac{1}{\sqrt{x^3+1}}<x^{-3/2}$ για $4 \le x \le 9$

$$\int_{-4}^{9}\frac{dx}{\sqrt{x^3+1}} \le \int_{-4}^{9} x^{-3/2}dx \Rightarrow \left[-2x^{-1/2}\right]_4^9=\frac{1}{3}$$

6.6 $\int_0^1 x^2 \sin x \, dx \le \int_0^1 x^2 dx=\frac{x^3}{3}\Big|_0^1=\frac{1}{3}$ διότι $\sin x \le 1$ στο $[0,1]$

6.7 (i) $\frac{\sqrt{1}+\sqrt{2}+\ldots+\sqrt{n}}{n^{3/2}}=\sum_{k=1}^{n}\frac{\sqrt{k}}{n^{3/2}}=\sum_{k=1}^{n}\frac{1}{n}\sqrt{\frac{k}{n}}=\sum_{k=1}^{n}f(x_k^*)\Delta_x$

όπου $f(x)=\sqrt{x}$, $x_k^*=\frac{k}{n}$ και $\Delta_x=\frac{1}{n}$

Άρα: $\lim_{n\to\infty}\sum_{k=1}^{n}f(x_k^*)\Delta_x=\int_0^1\sqrt{x}dx=\frac{2}{3}x^{3/2}\Big|_0^1=\frac{2}{3}$

(ii) $\frac{1^4+2^4+\ldots+n^4}{n^5}=\sum_{k=1}^{n}\frac{k^4}{n^5}=\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4\frac{1}{n}=\sum_{k=1}^{n}f(x_k^*)\Delta_x$

όπου $f(x)=x^4$, $x_k^*=\frac{k}{n}$ και $\Delta_x=\frac{1}{n}$

Άρα : $\lim_{n\to\infty}\sum_{k=1}^{n}f(x_k^*)\Delta_x=\int_0^1 x^4=\frac{x^5}{5}\Big|_0^1=\frac{1}{5}$

6.8 (i) $\int_1^x t^{-1/2}dt=3\Rightarrow 2t^{1/2}\Big|_1^x=3\Rightarrow 2\sqrt{x}-2=3\Rightarrow 2\sqrt{x}=5\Rightarrow\sqrt{x}=\frac{5}{2}\Rightarrow x=\frac{25}{4}$

(ii) $\int_x^0\frac{1}{(3t+1)^2}dt=-\frac{1}{6}\Rightarrow -\frac{1}{3}\frac{1}{(3t+1)}\Big|_x^0=-\frac{1}{6}\Rightarrow -\frac{1}{3}+\frac{1}{3}\frac{1}{(3x+1)}=-\frac{1}{6}\Rightarrow\frac{1}{3}\frac{1}{(3x+1)}=\frac{1}{6}$

$\Rightarrow 3x+1=2\Rightarrow x=\frac{1}{3}$

(iii)

$\int_2^x(4t-1)dt=9\Rightarrow\left(2t^2-t\right)\Big|_2^x=9\Rightarrow 2x^2-x-2(2)^2+2=9\Rightarrow 2x^2-x-15=0\Rightarrow x=-\frac{5}{2}$ και $x=3$

6.9 (i) $f(x)=2+|x|,[-3,1]$

$\bar{f}=\frac{1}{(1)-(-3)}\int_{-3}^{1}\left(2+|x|\right)dx=\frac{1}{4}\left\{\int_{-3}^{0}\left(2-x\right)dx+\int_0^1\left(2+x\right)dx\right\}=\frac{1}{4}\left[2x-\frac{x^2}{2}\right]_{-3}^{0}+\frac{1}{4}\left[2x+\frac{x^2}{2}\right]_0^1=\frac{13}{4}$

Από θεώρημα μέσης τιμής: $f\left(x^*\right)=\bar{f}\Rightarrow 2+\left|x^*\right|=\frac{13}{4}\Rightarrow\left|x^*\right|=\frac{5}{4}\Rightarrow x^*=\pm\frac{5}{4}$

Όμως: $\frac{5}{4}\notin[-3,1]$. Άρα $x^*=-\frac{5}{4}$

(ii) $f(x)=\sin^2x,[0,\pi]$

$\bar{f}=\frac{1}{\pi-0}\int_0^{\pi}\sin^2x\,dx=\frac{1}{\pi}\int_0^{\pi}\frac{1-\cos2x}{2}dx=\frac{1}{\pi}\left[\frac{1}{2}x-\frac{1}{4}\sin2x\right]_0^{\pi}=\frac{1}{2}$

Από θεώρημα μέσης τιμής: $f\left(x^*\right)=\bar{f}\Rightarrow\sin^2x^*=\frac{1}{2}\Rightarrow\sin x^*=\pm\frac{1}{\sqrt{2}}=\pm\frac{\sqrt{2}}{2}$

$\Rightarrow x^*=\frac{\pi}{4},\frac{3\pi}{4}\in[0,\pi]$

6.9 (iii) $f(x)=\frac{x}{\sqrt{x^2+9}}, [0,4]$

$$\bar{f}=\frac{1}{4-0}\int_0^4 \frac{x}{\sqrt{x^2+9}}dx=\frac{1}{4}\left[\sqrt{x^2+9}\right]_0^4=\frac{1}{2}$$

Από θεώρημα μέσης τιμής ολοκληρωμάτων: $f\left(x^*\right)=\bar{f} \Rightarrow \frac{x^*}{\sqrt{x^2+9}}=\frac{1}{2} \Rightarrow \frac{\left(x^*\right)^2}{\left(x^*\right)^2+9}=\frac{1}{4}$

$\Rightarrow 4\left(x^*\right)^2=\left(x^*\right)^2+9 \Rightarrow 3\left(x^*\right)^2=9 \Rightarrow \left(x^*\right)^2=3 \Rightarrow x^*=\pm\sqrt{3}$

Όμως: $-\sqrt{3} \notin [0,4]$. Άρα: $x^*=\sqrt{3}$

6.10 Θέτουμε $u=g(x)$ τότε:

$$\frac{d}{dx}\int_\alpha^{g(x)} f(t)dt=\frac{d}{dx}\int_\alpha^u f(t)dt=\frac{d}{du}\left[\int_\alpha^u f(t)dt\right]\frac{du}{dx}=f(u)g'(x)=f(g(x))g'(x)$$

Τώρα παρατηρούμε ότι ότι: $\int_{h(x)}^{g(x)} f(t)dt=\int_{h(x)}^{\alpha} f(t)dt+\int_\alpha^{g(x)} f(t)dt=\int_\alpha^{g(x)} f(t)dt-\int_\alpha^{h(x)} f(t)dt$

Άρα: $\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)dt=\frac{d}{dx}\int_\alpha^{g(x)} f(t)dt-\frac{d}{dx}\int_\alpha^{h(x)} f(t)dt=f(g(x))g'(x)-f(h(x))h'(x)$

6.11 (i) $\frac{d}{dx}\int_{x^2}^{x^3} \sin^2 t\, dt=3x^2\sin^2\left(x^3\right)-2x\sin^2\left(x^2\right)$

(ii) $\frac{d}{dx}\int_{-x}^{x} \frac{1}{1+t}dt=\frac{1}{1+x}-\frac{-1}{1-x}=\frac{2}{1-x^2}$

(iii) $\frac{d}{dx}\int_{-1}^{x^2+\sqrt{x}} \left(t+\sqrt{t}\right)dt=\left(2x+\frac{1}{2\sqrt{x}}\right)\left(x^2+\sqrt{x}+\sqrt{x^2+\sqrt{x}}\right)$

6.12 $f'(x)=\frac{1}{1+x^2}-\frac{1}{1+1/x^2}\left(-\frac{1}{x^2}\right)=0$

$\Rightarrow f(x)$ σταθερή στο $(0,+\infty)$

6.13 $x^2 \le f(x) \le 6, \forall x \in [-1,2] \Rightarrow \int_{-1}^2 x^2 dx \le \int_{-1}^2 f(x)dx \le \int_{-1}^2 6dx$

$\Rightarrow \frac{x^3}{3}\Big|_{-1}^2 \le \int_{-1}^2 f(x)dx \le 6x\Big|_{-1}^2 \Rightarrow 3 \le \int_{-1}^2 f(x)dx \le 18 \Rightarrow A=3, B=18$

6.14 $F(x)=\int_1^x \frac{1}{1+t^2}dt$

(i) $F(1)=\int_1^1 \frac{1}{1+t^2}dt=0$

(ii) $F'(x)=\frac{1}{1+x^2} \Rightarrow F'(1)=\frac{1}{1+1}=\frac{1}{2}$

(iii) $F(4)\text{-}F(2)=\int_1^4 \frac{1}{1+t^2}dt-\int_1^2 \frac{1}{1+t^2}dt=\int_2^4 \frac{1}{1+t^2}dt$

Όμως στο $[2,4]$ έχουμε $\frac{1}{1+t^2} \le \frac{1}{5}$

$\Rightarrow \int_2^4 \frac{1}{1+t^2}dt \le \int_2^4 \frac{1}{5}dt=\frac{t}{5}\Big|_2^4=\frac{4}{5}-\frac{2}{5}=\frac{2}{5} \Rightarrow F(4)\text{-}F(2) \le \frac{2}{5}$

6.15 Παρατηρούμε ότι :

$\frac{1}{b\text{-}\alpha}\int_\alpha^b \left(f(x)\text{-}\overline{f}\right)dx=\frac{1}{b\text{-}\alpha}\int_\alpha^b f(x)dx-\frac{1}{b\text{-}\alpha}\int_\alpha^b \overline{f}\,dx=\overline{f}-\frac{1}{b\text{-}\alpha}x\overline{f}\Big|_\alpha^b$

$=\overline{f}-\frac{b\text{-}\alpha}{b\text{-}\alpha}\overline{f}=\overline{f}-\overline{f}=0$

$\Rightarrow \int_\alpha^b \left(f(x)\text{-}\overline{f}\right)dx=0$