<u>ΛΥΣΕΙΣ ΑΣΚΗΣΕΩΝ</u>
<u>ΚΕΦΑΛΑΙΟ 10^{o}</u>

10.1 (i) $\int_0^{+\infty} xe^{-x^2}dx=\lim_{k\to+\infty}\int_0^k xe^{-x^2}dx=\lim_{k\to+\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^k=\lim_{k\to+\infty}\frac{1}{2}\left[-e^{-k^2}+1\right]=\frac{1}{2}$

(ii) $\int_2^{+\infty}\frac{1}{x\sqrt{\ln x}}dx=\lim_{k\to+\infty}\int_2^k\frac{dx}{x\sqrt{\ln x}}=\lim_{k\to+\infty}\left[2\sqrt{\ln x}\right]_2^k=\lim_{k\to+\infty}\left[2\sqrt{\ln k}-2\sqrt{\ln 2}\right]=+\infty$

(iii) $\int_0^{+\infty}\frac{e^{-x}}{1+e^{-2x}}dx=\lim_{k\to+\infty}\int_0^k\frac{e^{-x}}{1+e^{-2x}}=\lim_{k\to+\infty}\left[-\tan^{-1}\left(e^{-x}\right)\right]_0^k=\lim_{k\to+\infty}\left[-\tan^{-1}\left(e^{-k}\right)+\frac{\pi}{4}\right]=\frac{\pi}{4}$ (1)

$\int_{-\infty}^0\frac{e^{-x}}{1+e^{-2x}}dx=\lim_{k\to-\infty}\left[-\tan^{-1}\left(e^{-x}\right)\right]_k^0=\lim_{k\to+\infty}\left[-\frac{\pi}{4}+\tan^{-1}\left(e^{-k}\right)\right]=-\frac{\pi}{4}+\frac{\pi}{2}=\frac{\pi}{2}$ (2)

Από (1) και (2): $\int_{-\infty}^{+\infty}\frac{e^{-x}}{1+e^{-2x}}dx=\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$

(iv) $\int_0^9\frac{dx}{\sqrt{9-x}}=\lim_{k\to9^-}\int_0^k\frac{dx}{\sqrt{9-x}}=\lim_{k\to9^-}\left[-2\sqrt{9-x}\right]_0^k=\lim_{k\to9^-}2\left(-\sqrt{9-k}+3\right)=6$

(v) $\int_{-3}^1\frac{xdx}{\sqrt{9-x^2}}=\lim_{k\to-3^+}\int_k^1\frac{xdx}{\sqrt{9-x^2}}=\lim_{k\to-3^+}\left[-\sqrt{9-x^2}\right]_k^1=\lim_{k\to-9^+}2\left(-\sqrt{8}+\sqrt{9-k^2}\right)=-\sqrt{8}$

(vi) $\int_0^2\frac{dx}{(x-2)^{2/3}}=\lim_{k\to2}\int_0^k\frac{dx}{(x-2)^{2/3}}=\lim_{k\to2}3(x-2)^{1/3}\Big|_0^k=\lim_{k\to2}3\left((k-2)^{1/3}-(-2)^{1/3}\right)=3\sqrt[3]{2}$

(vii) $\int_0^{+\infty}\frac{dx}{x^2}=\int_0^1\frac{dx}{x^2}+\int_0^{+\infty}\frac{dx}{x^2}$

Όμως: $\int_0^1\frac{dx}{x^2}=\lim_{k\to0^+}\int_k^1\frac{dx}{x^2}=\lim_{k\to0^+}\left[-\frac{1}{x}\right]_k^1=\lim_{k\to0^+}\left(\frac{1}{k}-1\right)=+\infty$

Άρα δεν υπάρχει το: $\int_0^{+\infty}\frac{dx}{x^2}$

(viii) $\int_1^{+\infty}\frac{dx}{x\sqrt{x^2-1}}=\int_1^2\frac{dx}{x\sqrt{x^2-1}}+\int_2^{+\infty}\frac{dx}{x\sqrt{x^2-1}}$

$\int_1^2\frac{dx}{x\sqrt{x^2-1}}=\lim_{k\to1^+}\int_k^2\frac{dx}{x\sqrt{x^2-1}}=\lim_{k\to1^+}\sec^{-1}x\Big|_k^2=\lim_{k\to1^+}\left(\frac{\pi}{3}-\sec^{-1}k\right)=\frac{\pi}{3}$ (1)

$\int_2^{+\infty}\frac{dx}{x\sqrt{x^2-1}}=\lim_{k\to+\infty}\sec^{-1}x\Big|_2^k=\lim_{k\to+\infty}\left(\frac{\pi}{3}-\sec^{-1}k\right)=\frac{\pi}{2}-\frac{\pi}{3}$ (2)

Από (1) και (2): $\int_1^{+\infty}\frac{dx}{x\sqrt{x^2-1}}=\frac{\pi}{2}-\frac{\pi}{3}+\frac{\pi}{3}=\frac{\pi}{2}$

(ix) $\int_0^{+\infty}\frac{dx}{\sqrt{x}\,(x+4)}$

Θέτω : $\sqrt{x}=u\Rightarrow\frac{1}{2}\frac{1}{\sqrt{x}}dx=du,\ x=0\Rightarrow u=0$ και $x\to+\infty\Rightarrow u\to+\infty$

$\int_0^{+\infty}\frac{dx}{\sqrt{x}\,(x+4)}=2\int_0^{+\infty}\frac{du}{u^2+4}=2\lim_{k\to\infty}\left[\frac{1}{2}\tan^{-1}\frac{u}{2}\right]_0^k=\lim_{k\to\infty}\tan^{-1}\frac{k}{2}=\frac{\pi}{2}$

10.2 (i) $5=\int_0^{+\infty} e^{-\alpha x}dx=\lim_{k\to\infty}\int_0^k e^{-\alpha x}dx=\lim_{k\to\infty}\left[-\frac{e^{-\alpha x}}{\alpha}\right]_0^k=\lim_{k\to\infty}\left[-\frac{1}{\alpha}e^{-\alpha k}+\frac{1}{\alpha}\right]=\frac{1}{\alpha}\Rightarrow \alpha=\frac{1}{5}$

(ii) $1=\int_0^{+\infty}\frac{dx}{x^2+\alpha^2}=\lim_{k\to\infty}\int_0^k\frac{dx}{x^2+\alpha^2}=\lim_{k\to\infty}\frac{1}{\alpha}\tan^{-1}\left(\frac{x}{\alpha}\right)\Big|_0^k=\lim_{k\to\infty}\frac{1}{\alpha}\left(\tan^{-1}\frac{k}{\alpha}-\tan^{-1}0\right)$

$=\frac{1}{\alpha}\frac{\pi}{2}\Rightarrow \alpha=\frac{\pi}{2}$

10.3 $\int_0^{+\infty}e^{-x^2}dx=\frac{1}{2}\sqrt{\pi}$

(i) $\int_0^{+\infty}\frac{e^{-x}}{\sqrt{x}}dx$

Θέτω : $u=\sqrt{x}\Rightarrow du=\frac{1}{2\sqrt{x}}dx$, $x=0\Rightarrow u=0$ και $x\to+\infty\Rightarrow u\to+\infty$

$\int_0^{+\infty}\frac{e^{-x}}{\sqrt{x}}dx=2\int_0^{+\infty}e^{-u^2}dx=2\frac{\sqrt{\pi}}{2}=\sqrt{\pi}$

(ii) $\int_0^{+\infty}e^{-\alpha^2x^2}dx$

Θέτω : $u=\alpha x\Rightarrow du=\alpha dx$, $x=0, u=0$ και $x\to+\infty, u\to+\infty$

$\int_0^{+\infty}e^{-\alpha^2x^2}dx=\frac{1}{\alpha}\int_0^{+\infty}e^{-u^2}du=\frac{\sqrt{\pi}}{2\alpha}$

10.4 (i) $\lim_{x\to0}\frac{2\cosh x-2}{1-\cos 2x}=\lim_{x\to0}\frac{2\sinh x}{1+2\sin 2x}$

(ii) $\lim_{x\to0}\frac{1-e^{-2x}}{x^2+3x}=\lim_{x\to0}\frac{2e^{-2x}}{2x+3}=\frac{2}{3}$

(iii) $\lim_{x\to2}\frac{\ln(5x-9)}{x^3-8}=\lim_{x\to2}\frac{\frac{5}{5x-9}}{3x^2}=\frac{5}{12}$

(iv) $\lim_{x\to0}\frac{x-\tan^{-1}x}{x^3}=\lim_{x\to0}\frac{1-\frac{1}{1+x^2}}{3x^2}=\lim_{x\to0}\frac{\frac{1+x^2-1}{1+x^2}}{3x^2}=\lim_{x\to0}\frac{x^2}{3x^2\left(1+x^2\right)}=\lim_{x\to0}\frac{1}{3\left(1+x^2\right)}=\frac{1}{3}$

(v) $\lim_{x\to-1}\frac{x^2-1}{\ln(3x+4)}=\lim_{x\to-1}\frac{2x}{\frac{3}{3x+4}}=-\frac{2}{3}$

(vi) $\lim_{x\to\pi}\frac{\sin^2x}{1+\cos 3x}=\lim_{x\to\pi}\frac{2\sin x\cos x}{-3\sin 3x}=\lim_{x\to\pi}\frac{2\cos^2x-2\sin^2x}{-9\cos 3x}=\lim_{x\to\pi}\frac{2\cos 2x}{-9\cos 3x}=\frac{2}{9}$

10.5 $\lim_{x\to 0}\frac{\alpha+\cos bx}{x^2}=-4$

Υποχρεωτικά το $\alpha=-1$ διότι διαφορετικά το όριο κάνει $+\infty$. Έτσι έχουμε:

$$-4=\lim_{x\to 0}\frac{-1+\cos bx}{x^2}=\lim_{x\to 0}\frac{-b\sin bx}{2x}=\lim_{x\to 0}\frac{-b^2\cos bx}{2}=-\frac{b^2}{2}\Rightarrow b^2=8\Rightarrow b=\pm 2\sqrt{2}$$

10.6 (i) $$\lim_{x\to 0}\frac{(2+x)\ln(1-x)}{\left(1-e^x\right)\cos x}=\lim_{x\to 0}\frac{\ln(1-x)-(2+x)\frac{1}{1-x}}{-e^x\cos x-\left(1-e^x\right)\sin x}=\frac{-2}{-1}=2$$

(ii) $$\lim_{x\to+\infty}x\ln\left(\frac{x+1}{x-1}\right)=\lim_{x\to+\infty}\left[\frac{\ln(x+1)}{1/x}-\frac{\ln(x-1)}{1/x}\right]=\lim_{x\to+\infty}\left[\frac{\frac{1}{x+1}}{-1/x^2}-\frac{\frac{1}{x-1}}{-1/x^2}\right]$$

$$=\lim_{x\to+\infty}\left[-x^2\left[\frac{x-1-x-1}{(x+1)(x-1)}\right]\right]=\lim_{x\to+\infty}\frac{2x^2}{x^2-1}=2$$

10.7 (i) $\int_0^1\ln x\,dx$

Υπολογίζω πρώτα: $\int\ln x\,dx=x\ln x-\int\frac{1}{x}x\,dx=x\ln x-x$

$$\Rightarrow\int_0^1\ln x\,dx=\lim_{k\to 0^+}\int_k^1\ln x\,dx=\lim_{k\to 0^+}\left[x\ln x-x\right]_k^1=\lim_{k\to 0^+}\left[-1-k\ln k+k\right]=-1-\lim_{k\to 0^+}\frac{\ln k}{1/k}$$

$$=-1-\lim_{k\to 0^+}\frac{1/k}{-1/k^2}=-1+\lim_{k\to 0^+}k=-1$$

(ii) $\int_1^{+\infty}\frac{\ln x}{x^2}dx$

Υπολογίζω πρώτα: $\int\frac{\ln x}{x^2}dx=-\frac{1}{x}\ln x+\int\frac{1}{x^2}dx=-\frac{1}{x}\ln x-\frac{1}{x}$

$$\Rightarrow\int_1^{+\infty}\frac{\ln x}{x^2}dx=\lim_{k\to+\infty}\int_1^k\frac{\ln x}{x^2}dx=\lim_{k\to+\infty}\left[-\frac{1}{x}\ln x-\frac{1}{x}\right]_1^k=\lim_{k\to+\infty}\left[-\frac{1}{k}\ln k-\frac{1}{k}+1\right]$$

$$=1-\lim_{k\to+\infty}\frac{\ln k}{k}=1-\lim_{k\to+\infty}\frac{1/k}{1}=1+0=0$$

(iii) $\int_0^{+\infty}xe^{-3x}dx$

Υπολογίζω πρώτα: $\int xe^{-3x}dx=-\frac{1}{3}xe^{-3x}+\frac{1}{3}\int e^{-3x}dx=-\frac{1}{3}xe^{-3x}-\frac{1}{9}e^{-3x}$

$$\Rightarrow\int_0^{+\infty}xe^{-3x}dx=\lim_{k\to+\infty}\int_0^k xe^{-3x}dx=\lim_{k\to+\infty}\left[-\frac{1}{3}xe^{-3x}-\frac{1}{9}e^{-3x}\right]_0^k=\lim_{k\to+\infty}\left[-\frac{1}{3}ke^{-3k}-\frac{1}{9}e^{-3k}+\frac{1}{9}\right]$$

$$=\frac{1}{9}+0-\frac{1}{3}\lim_{k\to+\infty}\frac{k}{e^{3k}}=\frac{1}{9}-\frac{1}{3}\lim_{k\to+\infty}\frac{1}{3e^{3k}}=\frac{1}{9}$$

10.8 (i) Έστω : $y=\left(1+2x\right)^{-3/x} \Rightarrow \ln y= -\frac{3}{x}\ln(1+2x) \Rightarrow \lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{-3\ln(1+2x)}{x}$

$= -3\lim_{x\to\infty}\frac{\frac{2}{1+2x}}{1} = -6 \Rightarrow \lim_{x\to\infty} y=e^{-6}$

(ii) Έστω : $y=\left(\cos\left(2/x\right)\right)^{x^2} \Rightarrow \ln y= x^2\ln\left(\cos\left(2/x\right)\right) \Rightarrow \lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln\left(\cos\left(2/x\right)\right)}{1/x^2}$

$\lim_{x\to\infty}\frac{\frac{2}{x^2}\frac{\sin\left(2/x\right)}{\cos\left(2/x\right)}}{-2\,1/x^3} = \lim_{x\to\infty}\frac{-\tan\left(2/x\right)}{1/x} = \lim_{x\to\infty}\frac{2/x^2\sec^2\left(2/x\right)}{-1/x^2} = -2\lim_{x\to\infty}\frac{1}{\cos^2\left(2/x\right)}$

$\Rightarrow \lim_{x\to\infty} y=e^{-2}$

(iii) Έστω : $y=\left(e^{2x}-1\right)^{1/\ln x} \Rightarrow \ln y=\frac{1}{\ln x}\ln\left(e^{2x}-1\right) \Rightarrow \lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln\left(e^{2x}-1\right)}{\ln x}$

$=\lim_{x\to 0^+}\frac{\frac{2}{e^{2x}-1}}{1/x} = \lim_{x\to 0^+}\frac{2x}{e^{2x}-1} = \lim_{x\to 0^+}\frac{2}{2e^{2x}} = 1 \Rightarrow \lim_{x\to 0^+} y=e$

(iv) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to 0}\frac{e^x-1-x}{xe^x-x}=\lim_{x\to 0}\frac{e^x-1}{xe^x+e^x-1}=\lim_{x\to 0}\frac{e^x}{xe^x+2e^x}=\frac{1}{2}$

(v) $\lim_{x\to\infty}\left[x-\ln\left(1+2e^x\right)\right]=\lim_{x\to\infty}\left[\ln e^x-\ln\left(1+2e^x\right)\right]2$

$=\lim_{x\to\infty}\ln\left(\frac{e^x}{1+2e^x}\right)=\lim_{x\to\infty}\ln\left(\frac{1}{e^{-x}+2}\right)=\ln\frac{1}{2}=-\ln 2$

(vi) $\lim_{x\to \pi/2^-}\frac{4\tan x}{1+\sec x}=\lim_{x\to \pi/2^-}\frac{4\sec^2 x}{\sec x\tan x}=\lim_{x\to \pi/2^-}\frac{4}{\sin x}=4$

10.9 $\sqrt{1+t^3}\ge t^{3/2}, \forall t\ge 0 \Rightarrow \int_0^{+\infty}\sqrt{1+t^3}dt \ge \int_0^{+\infty}t^{3/2}dt$

$\int_0^{+\infty}t^{3/2}dt=\lim_{k\to\infty}\int_0^k t^{3/2}dt=\lim_{k\to\infty}\frac{2}{5}t^{5/2}\Big|_0^k=\lim_{k\to\infty}\left[\frac{2}{5}k^{5/2}\right]=+\infty$

Άρα : $\int_0^{+\infty}\sqrt{1+t^3}dt=+\infty$

$\lim_{x\to\infty}\frac{\int_0^{2x}\sqrt{1+t^3}dt}{x^{5/2}}=\lim_{x\to\infty}\frac{2\sqrt{1+8x^3}}{5/2\,x^{3/2}}=\frac{2}{5}\lim_{x\to\infty}\sqrt{\frac{1+8x^3}{x^3}}=\frac{4}{5}\sqrt{8}=\frac{8\sqrt{2}}{5}$

10.10 Έχουμε ότι: $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} f'(x)=0$ και $\lim_{x\to 0^+} f''(x)=\alpha$

Θέτω: $u=\frac{t}{\sqrt{x}} \Rightarrow x=\frac{t^2}{u^2}$, $x \to +\infty \Rightarrow u \to 0$

Άρα: $\lim_{x\to\infty}\left[x f\left(\frac{t}{\sqrt{x}}\right)\right]=\lim_{x\to\infty}\frac{t^2}{u^2}f(u)=t^2\lim_{x\to\infty}\frac{f'(u)}{2u}=t^2\lim_{x\to\infty}\frac{f''(u)}{2}=\frac{t^2 f''(0)}{2}=\frac{t^2\alpha}{2}$

10.11(i) $\int_0^{+\infty}\frac{dx}{x^5}=\int_0^1\frac{dx}{x^5}+\int_1^{+\infty}\frac{dx}{x^5}$

Όμως: $\int_0^1\frac{dx}{x^5}=\lim_{k\to 0^+}\int_k^1\frac{dx}{x^5}=\lim_{k\to 0^+}\left.\frac{x^{-4}}{-4}\right|_k^1=\lim_{k\to 0^+}\left(-\frac{1}{4}+\frac{k^{-4}}{4}\right)=-\frac{1}{4}+\lim_{k\to 0^+}\frac{1}{k^4}=+\infty$

Άρα δεν υπάρχει το $\int_0^{+\infty}\frac{dx}{x^5}$

(ii) Υπολογίζω πρώτα το $\int\sqrt{x}\ln x dx$

$\int\sqrt{x}\ln x dx=\frac{2}{3}x^{3/2}\ln x-\frac{2}{3}\int x^{3/2}\frac{1}{x}dx=\frac{2}{3}x^{3/2}\ln x-\frac{2}{3}\int x^{1/2}dx=\frac{2}{3}x^{3/2}\ln x-\frac{4}{9}x^{3/2}$

$\int_0^1\sqrt{x}\ln x dx=\lim_{k\to 0^+}\left[\frac{2}{3}x^{3/2}\ln x-\frac{4}{9}x^{3/2}\right]_1^k=\lim_{k\to 0^+}\left[-\frac{4}{9}-\frac{2k^{3/2}\ln k}{3}+\frac{4}{9}k^{3/2}\right]=-\frac{4}{9}-\frac{2}{3}\lim_{k\to 0^+}\frac{\ln k}{1/k^{3/2}}$

$=-\frac{4}{9}-\frac{2}{3}\lim_{k\to 0^+}\frac{1/k}{-\frac{3}{2}k^{-5/2}}=-\frac{4}{9}-\frac{2}{3}\lim_{k\to 0^+}\frac{2}{3}k^{3/2}=-\frac{4}{9}$

(iii) Υπολογίζω πρώτα το $\int\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx=\int -2e^{-\sqrt{x}}d\left(-\sqrt{x}\right)=-2e^{-\sqrt{x}}$

$\int_0^4\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx=\lim_{k\to 0^+}\left[-2e^{-\sqrt{x}}\right]_k^4=\lim_{k\to 0^+}\left(-2e^{-2}+2e^{-\sqrt{k}}\right)=-2e^{-2}+2=2\left(1-e^{-2}\right)$

10.12 $y=\alpha^x \Rightarrow y'=\alpha^x\ln x$

(i) $\lim_{x\to 0}\frac{9^x-3^x}{x}=\lim_{x\to 0}\frac{9^x\ln 9-3^x\ln 3}{1}=\ln 9-\ln 3=\ln\frac{9}{3}=\ln 3$

(ii) $\lim_{x\to 0}\frac{\int_0^x\sin\left(t^2\right)dt}{\sin\left(x^2\right)}=\lim_{x\to 0}\frac{\sin\left(x^2\right)}{2x\cos\left(x^2\right)}=\lim_{x\to 0}\frac{2x\cos\left(x^2\right)}{-4x^2\sin\left(x^2\right)+2\cos\left(x^2\right)}=\frac{0}{0+2}=0$

(iii) $\lim_{x\to +\infty}\left(\frac{x}{x-3}\right)^x$

Έστω: $y=\left(\frac{x}{x-3}\right)^x \Rightarrow \ln y=x\ln\left(\frac{x}{x-3}\right)\Rightarrow \lim_{x\to +\infty}\left(\ln y\right)=\lim_{x\to\infty}\frac{\ln\left(x\right)-\ln\left(x-3\right)}{1/x}$

$=\lim_{x\to\infty}\frac{\frac{1}{x}-\frac{1}{x-3}}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{\frac{x-3-x}{x\left(x-3\right)}}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{3x}{x-3}=\lim_{x\to\infty}\frac{3}{1-3/x}=3\Rightarrow\lim_{x\to\infty}y=e^3$

10.13 $E=\int_0^1 x^{-1/3}dx=\lim_{k\to 0^+}\int_k^1 x^{-1/3}dx=\lim_{k\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_k^1=\lim_{k\to 0^+}\left[\frac{3}{2}-\frac{3}{2}k^{2/3}\right]=\frac{3}{2}$

$V=\pi\int_0^1\left(x^{-1/3}\right)^2 dx=\pi\int_0^1 x^{-1/3}dx=\lim_{x\to 0^+}\pi\int_k^1 x^{-1/3}dx=\pi\lim_{x\to 0^+}\left[3x^{2/3}\right]_k^1=\pi\lim_{x\to 0^+}\left[3-3k^{2/3}\right]=3\pi$